25. Consider the following statements :
   1. A linear programming problem with three variables and two constraints can be solved by graphical methd.
   2. For solutions of a linear programming problem with mixed constraints, Big-M method can be employed.
   3. In the solution process of a linear programming problem using Big-M method when an artificial variable leaves the basis, the column of the artificial variable can be removed from all subsequent tables.

   Which of these statements are correct ?
   (a) 1, 2 and 3. (b) 1 and 2. (c) 1 and 3. (d) 2 and 3.
   [IES (Mech.) 2000]

26. Consider the following statements regarding linear programming :
   1. Dual of a dual is primal.
   2. When two minimum ratios of the right hand side to the coefficient in the key column are equal, degeneacy may take place.
   3. When an artificial variable leaves the basis, its column can be deleted from the subsequent simplex tables.

   Select the correct answer from the codes given below :
   codes :
   (a) 1, 2 and 3. (b) 1 and 2. (c) 2 and 3. (d) 1 and 3.
   [IES (Mech.) 2001]

27. In the solution linear programming problems by Simplex Method, for deciding the leaving variable
   (a) The maximum negative coefficient in the objective function row is selected.
   (b) The minimum positive ratio of the right hand side to the first decision variable is selected.
   (c) the maximum positive ratio of the right hand side to the coefficients in the key column is selected.
   (d) The minimum positive ratio of the right hand side to the coefficient in the key column is selected. [IES 2003]

Answers

1. (b)	2. (d)	3. (b)	4. (d)	5. (c)	6. (d)	7. (d)	8. (c)	9. (c)
10. (d)	11. (c)	12. (c)	13. (a)	14. (c)	15. (c)	16. (b)	17. (a)	18. (a)
19. (b)	20. (c)	21. (a)	22. (d)	23. (c)	24. (d)	25. (d)	26. (a)	27. (d).

✱✱✱

6

TRANSPORTATION MODEL

6.1 INTRODUCTION

As already defined and discussed earlier, the simplex procedure can be regarded as the most generalized method for linear programming problems. However, there is very interesting class of *'Allocation Methods'* which is applied to a lot of very practical problems generally called *'Transportation Problems'*. Whenever it is possible to place the given linear programming problem in the transportation frame-work, it is far more simple to solve it by *'Transportation Technique'* than by *'Simplex'*.

Let the nature of transportation problem be examined first. If there are more than one centres, called *'origins'*, from where the goods need to be shipped to more than one places called *'destinations'* and the costs of shipping from each of the *origins* to each of the *destinations* being different and known, the problem is to ship the goods from various *origins* to different *destinations* in such a manner that the cost of shipping or transportation is minimum.

Thus, we can formally define the transportation problem as follows :

Definition. ***The Transportation Problem*** *is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.*

For example, a tyre manufacturing concern has m factories located in m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

The various features of linear programming can be observed in these problems. Here the availability as well as the requirements of the various centres are finite and constitute the limited resources. It is also assumed that the cost of shipping is linear (for example, the costs of shipping of *two* objects will be *twice* that of shipping a *single* object). However, this condition is not often true in practical problems, but will have to be assumed so that the linear programming technique may be applicable to such problems. These problems thus could also be solved by 'Simplex'. Mathematically, the problem may be stated as given in the following section.

6.2 MATHEMATICAL FORMULATION

Let there be m origins, ith origin possessing a_i units of a certain product, whereas there are n destinations (n may or may not be equal to m) with destination j requiring b_j units. Costs of shipping of an item from each of m origins (sources) to each of the n destinations are known either directly or indirectly in terms of mileage, shipping hours, etc. Let c_{ij} be the cost of shipping one unit product from ith origin (source) to jth destination., and 'x_{ij}' be the amount to be shipped from ith origin to jth destination.

It is also assumed that total availabilities Σa_i satisfy the total requirements Σb_j, *i.e.*,

$$\Sigma a_i = \Sigma b_j \quad (i = 1, 2, \ldots, m ; j = 1, 2, \ldots, n) \qquad \ldots(6\cdot1)$$

(***In case,*** $\Sigma a_i \neq \Sigma b_j$ some manipulation is required to make $\Sigma a_i = \Sigma b_j$, which will be shown later).

The problem now is to determine non-negative (≥ 0) values of 'x_{ij}' satisfying both, the availability constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, \dots, m \qquad \dots(6{\cdot}2)$$

as well as the requirement constraints :

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{for } j = 1, 2, \dots, n \qquad \dots(6{\cdot}3)$$

and *minimizing* the total cost of transportation (shipping)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij} \qquad (\textit{objective function}). \qquad \dots(6{\cdot}4)$$

It may be observed that the constraint equations (6·2), (6·3) and the objective function (6·4) are all linear in x_{ij}, so it may be looked like a linear programming problem.

This special type of LPP will be called a ***Transportation Problem*** (T.P.).

Remark : By requiring strict inequalities $a_i > 0$ and $b_j > 0$ we are not restricting anything. Since all $x_{ij} \geq 0$, it follows that each $a_i \geq 0$ and each $b_j \geq 0$. Moreover any $a_k = 0 \Rightarrow x_{kj} = 0$ and thus can be eliminated from the problem.

Q. 1. Explain Transportation problem and show that it can be considered as L.P.P.
2. Formulate transportation problem as a L.P.P.
3. Specify a transportation problem (TP). Is this an LPP ? **[AIMS (Bangalore) MBA 2002]**
4. What is a Transportation Problem ? **[Meerut 2002; IGNOU 99, 96]**
5. Give a mathematical formulation of the transportation and the simplex methods. What are the differences in the nature of problems that can be solved by these methods.
6. What is meant by a classical transportation problem ? What is its mathematical formulation ? Explain briefly your symbols.

6.3 MATRIX FORM OF TRANSPORTATION PROBLEM

Consider the transportation problem as mathematically formulated above. The set of constraints $\sum_{j=1}^{n} x_{ij} = a_i$ $(i = 1, 2, \dots, m)$ and $\sum_{i=1}^{m} x_{ij} = b_j$ $(j = 1, 2, \dots n)$ represent $m + n$ equations in mn non-negative variables x_{ij}. Each variable x_{ij} appears in exactly two constraints, one associated with the ith origin O_i and the other with the jth destination D_j. In the above ordering of constraints, first we write the origin-equations and then destination-equations. Then the transportation problem can be restated in the matrix form as :

Minimize $z = \mathbf{CX}$, $\mathbf{X} \in \mathbf{R}^{mn}$, subject to the constraints $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, $\mathbf{b} \in \mathbf{R}^{m+n}$

where $\mathbf{X} = [x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]$, $\mathbf{C}$ *is the cost vector,* $\mathbf{b} = [a_1, a_2, \dots, a_m, b_1, b_2, \dots,$ $\dots, b_n]$ and $\mathbf{A}$ is an $(m + n) \times mn$ real matrix containing the coefficients of constraints.

It is worthnoting that the elements of $\mathbf{A}$ are either 0 or 1. Thus the *general* LPP can be reduced to *transportation problem* if

(i) a_{ij}'s are restricted to the values 0 and 1; and (ii) the units among the constraints are homogeneous.

For example, if $m = 2$, $n = 3$, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{23}^{(1)} & \mathbf{e}_{23}^{(2)} \\ \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}$$

And, therefore, for a general transportation problem, we may write

$$\mathbf{A} = \begin{bmatrix} \mathbf{e}_{mn}^{(1)} & \mathbf{e}_{mn}^{(2)} & \dots & \mathbf{e}_{mn}^{(m)} \\ \mathbf{I}_n & \mathbf{I}_n & \dots & \mathbf{I}_n \end{bmatrix}$$

where $\mathbf{e}_{mn}^{(i)}$ is an $m \times n$ matrix having a row of unit elements as its ith row and 0's everywhere else, and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

If $\mathbf{a}_{ij}$ denotes the column vector of A associated with any variable x_{ij}, then it can be easily verified that

$$\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}\ , \text{ where } \mathbf{e}_i\ , \mathbf{e}_{m+j} \in \mathbf{R}^{m+n} \text{ are unit vectors.}$$

Q. 1. Show how a 2 × 3 transportation problem may be transformed into a special network termed bipartite network.

2. If a transportation porblem has p factories and 2 retail shops, what is the number of variables and what is the number of constraints ? **[IGNOU 99, 96]**

6.4 FEASIBLE SOLUTION, BASIC FEASIBLE SOLUTION AND OPTIMUM SOLUTION

The terms *feasible solution, basic feasible solution* and *optimum solution* may be formally defined with reference to the transportation problem (T.P.) as follows :

(i) **Feasible Solution (FS).** A set of non-negative individual allocations ($x_{ij} \ge 0$) which simultaneously removes deficiencies is called a ***feasible solution.***

(ii) **Basic Feasible Solution (BFS).** A feasible solution to a m-origin, n- destination problem is said to be *basic* if the number of positive allocations are $m + n - 1$, *i.e.*, one less than the sum of rows and columns (*see Theorem* 6·2).

If the number of allocations in a basic feasible solution are less than $m + n - 1$, it is called *degenerate* BFS (otherwise, ***non-degenerate*** BFS).

(iii) **Optimum Solution.** A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

Q. Define the terms (a) Feasible solution (b) Basic feasible solution (c) Optimum in solution.

6·4–1 Existence of Feasible Solution

Theorem 6.1. (***Existence of Feasible Solution***). *A necessary and sufficient condition for the existence of feasible solution of a transportation problem is* $\Sigma\, a_i = \Sigma\, b_j\ (i = 1, \ldots, m\ ; j = 1, \ldots, n)$. **[Rewa (M.P.) 93]**

Proof. ***The condition is necessary.*** Let there exist a feasible solution to the transportation problem. Then,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i\ ,\ \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \Leftrightarrow \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The condition is sufficient. Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k$ (say).

If $\lambda_i \ne 0$ be any real number such that $x_{ij} = \lambda_i\, b_j$ for all i and j , then λ_i is given by

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \lambda_i\, b_j = \lambda_i \sum_{j=1}^{n} b_j = k\,\lambda_i \Rightarrow \lambda_i = \frac{1}{k}\sum_{j=1}^{n} x_{ij} = \frac{a_i}{k}.$$

Thus, $\quad x_{ij} = \lambda_i\, b_j = \dfrac{a_i\, b_j}{k} \ge 0$, since $a_i > 0, b_j > 0$ for all i and j

Hence a feasible solution exists.

6·4–2 Basic Feasible Solution of Transportation Problem

It has been observed that a *transportation problem* is a special case of a *linear programming problem.* So a basic feasible solution of a transportation problem has the same definition as earlier given for L.P.P. (in *Sec* **4·8** , page 91). However, we observe that in the case of a T.P., there are only $m + n - 1$ basic variables out of mn unknowns. This happens due to redundancy in the constraints of the transportation problem. This can be easily justified by proving the following theorem.

Theorem 6.2. *The number of basic variables in a transportation problem are at the most* $m + n - 1$.

Proof. To prove this, consider the first $m + n - 1$ constraints of the transportation problem as

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, ..., n-1 \qquad \textbf{(n – 1 equations)} \qquad ...(1)$$

and $$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, .. m. \qquad \textbf{(m equations)} \qquad ...(2)$$

Now adding $(n-1)$ destination-constraints (1), we get

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \qquad ...(3)$$

Also, adding m origin-constraints (2), we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \qquad ...(4)$$

Then, subtracting (3) from (4), we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j \qquad ...(5)$$

or $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j \left(\because \sum_i a_i = \sum_j b_j \right)$$

or $$\sum_{i=1}^{m} \left(x_{in} + \sum_{j=1}^{n-1} x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = b_n + \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n-1} b_j$$

or $\sum_{i=1}^{m} x_{in} = b_n$ (which is exactly the last (nth) destination-constraint)

This obviously indicates that if the first $m + n - 1$ constraints are satisfied then $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ ensures that the $(m + n)$th constraint will be automatically satisfied.

Thus, out of $m + n$ equations, one (any) is redundant and remaining $m + n - 1$ equations form a linearly independent set. Hence the theorem is proved.

It is concluded that ***a basic feasible solution will consist of at most m + n – 1 positive variables, others being zero. In the degenerate case, some of the basic variables will also be zero, i.e., the number of positive variables will now become less than m + n – 1. By fundamental theorem of linear programming, one of the basic feasible solutions will be the optimal solution.***

6·4-3 Existence of Optimal Solution

Theorem 6.3. (Existence of an optimal solution). *There always exists an optimal solution to a balanced transportation problem.*

Proof. Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, so that a feasible solution x_{ij} exists. It, therefore, follows from the constraints of the problem that each x_{ij} is bounded, *viz.*, $0 \le x_{ij} \le \min(a_i, b_j)$.

Thus the feasible region of the problem is *closed, bounded* and *non-empty*, and hence there exists an optimal solution.

Note. In future discussion we shall assume that the above condition holds for the transporta*tion problem* without mentioning it.

Q. 1. Prove that the transportation problem always possesses a feasible solution.

2. If all the sources are emptied and all the destinations are filled, show that $\Sigma a_i = \Sigma b_j$ is a necessary and sufficient condition for the existence of a feasible solution to the transportation problem. **[Delhi B.Sc. (Maths.) 90]**

3. Prove that the solution of the transportation problem is invariant under the addition (subtraction) of the same constant to (from) any row or column of the unit cost matrix of the problem.

4. Derive a mathematical model for a cost-minimizing 'Transportation Problem'. Show that every transportation problem has a feasible solution.

6.5 TABULAR REPRESENTATION

Suppose there are m factories and n warehouses. The transportation problem is usually represented in a tabular form (**Table 6·1**). Calculations are made directly on the 'transportation arrays' which give the current trial solution.

Table 6·1

Warehouse → Factory ↓	W_1	W_2	...	W_j	...	W_n	Factory Capacities
F_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1
F_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}	a_2
:	:	:		:		:	:
F_i	c_{i1}	...c_{i2}	...	c_{ij}	...	c_{in}	a_i
:	:	:	:	:	:	:	:
F_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m
Warehouse requirements	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Table 6·2

Warehouse → Factory ↓	W_1	W_2	...	W_j	...	W_n	Factory Capacities
F_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1
F_2	x_{21}	x_{22}	...	x_{2j}	..	x_{2n}	a_2
:	:	:		:		:	:
F_i	x_{i1}	...x_{i2}	...	x_{ij}	...	x_{in}	a_i
:	:	:	:	:	:	:	:
F_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m
Warehouse requirements	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

In general, **Tables 6·1 and 6·2** are combined by inserting each unit cost c_{ij} together with the corresponding amount x_{ij} into the cell (i, j). The product $x_{ij}\ (c_{ij})$ gives the net cost of shipping x_{ij} units from factory F_i to warehouse W_j.

Note. Whenever the amount x_{ij} and the corresponding unit cost c_{ij} are entered in the cell (i, j), there may be a confusion to distinguish between them. Therefore, in order to remove such confusion the quantities in parenthesis will denote the unit cost c_{ij}.

Q. 1. Describe the transportation table.

2. Describe the matrix form of the transportation problem. Illustrate with 2 origins and 3 destinations.

6.6 SPECIAL STRUCTURE OF TRANSPORTATION PROBLEM

The transportation problem has a *triangular basis, i.e.* the system of equations is represented in terms of basic variables only; non-basic variables are considered to be zero. The matrix of coefficients of the variables is triangular. In other words, there is an equation in which only one basic variable occurs ; in a second equation not more than two basic variables occur, in a third equation not more than three basic variables occur, and so on.

Equations (6·2) and (6·3) may be called the row and column equations, respectively.

Theorem 6.4. *The transportation problem has a triangular basis.*

Proof. To prove this theorem, consider equations (6·2) and (6·3) written row-wise and column-wise in the tabular form (**Table 6·2**).

There cannot be an equation in which no basic variable exists, otherwise the equation cannot be satisfied, for $a_i \neq 0$ or $b_j \neq 0$. The theorem will be proved by contradiction.

Suppose every equation has at least two basic variables. Then there will be at least two basic variables in each row, and the total number of basic variables will be at least $2m$. Also each column equation will have at least two basic variables, and hence in all there will be at least $2n$ variables. Let the total number of basic variables be N. Thus, $N \geq 2m$, $N \geq 2n$. Now, three cases may arise.

Case 1. If $m > n$, then $N \geq 2m$ becomes $N > m + n$.

Case 2. If $m < n$, then $N < 2n$ becomes $N < n + m$.

Case 3. If $m = n$, then $N \geq 2m$ becoems $N = m + n$.

Thus, it is observed that in every case $N \geq m + n$. But $N = m + n - 1$, which is a contradiction. Thus, the assumption of existing at least two basic variables in each row and each column is wrong. Therefore, at least one such row or column equation exists having one basic variable only.

Let x_{rc} be the only variable in the rth row and the cth column. Then, $x_{rc} = a_r$. Then equation can be eliminated from the system by deleting the rth row equation and substituting $x_{rc} = a_r$ in the cth column equation. Thus, rth row now stands cancelled, and b_c is replaced by $b'_c = b_c - a_r$.

The resulting system now consists of $m - 1$ row equations and n column equations of which $m + n - 2$ are linearly independent. Thus, there are $m + n - 2$ basic variables in this system. Repeat the process and it is concluded that there is an equation in the reduced system which has only one basic variable. If this equation happens to be the cth column equation in the original system, the cth column equation now contains two basic variables. Thus the original system has an equation which has at most two basic variables. Continue this process and ultimately it can be shown that there is an equation which has at most three basic variables, and so on.

Thus the theorem is now completely proved.

Q. 1. Prove that there are only $m + n - 1$ independent equations in a transportation problem, m and n being the number of origins and destinations, and any one equation can be dropped as the redundant equation.

2. What do you mean by the triangular form of a system of linear equations ? When we can say that a system of linear equations has a triangular basis.
3. Show that all bases for transportation problem are triangular.
4. What do you mean by non-degenerate basic feasible solution of a transportation problem.
5. State a transportation problem. When does it have a unique solution ? Explain.

6.7 LOOPS IN TRANSPORTATION TABLE AND THEIR PROPERTIES

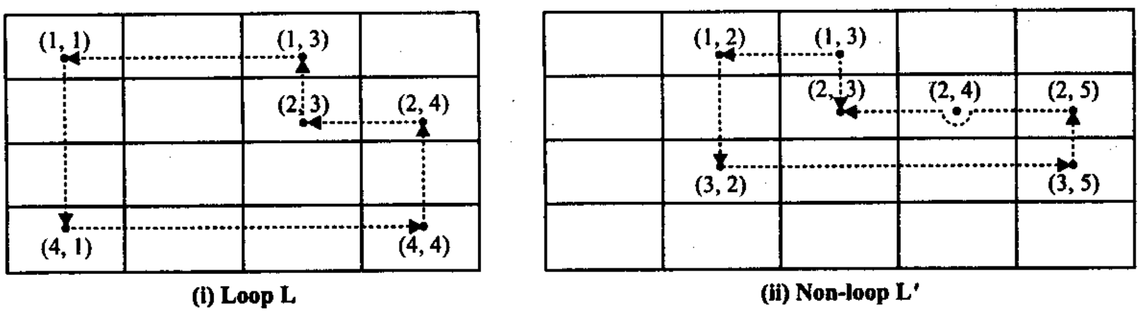
Loop. Def. *In a transportation table, an ordered set of four or more cells is said to form a* LOOP if,

(i) any two adjacent cells in the ordered set lie either in the same row or in the same column ; and

(ii) any three or more adjacent cells in the ordered set do not lie in the same row or in the same column.

The first cell of the set is considered to follow the last one in the set.

If we join the cells of a loop by horizontal and vertical lineup segments, we get a closed path satisfying the above conditions *(i)* and *(ii)*. Let us denote the (i, j)th cell of the transportation table by (i, j). Then it can be observed from the diagramatic illustration in Fig. 6.1, that the set $L = \{(1, 1), (4, 1), (4, 4), (2, 4), (2, 3), (1, 3)\}$ form a loop; and on the other hand, the set $L' = \{(3, 2), (3, 5), (2, 5), (2, 4), (2, 3), (1, 3), (1, 2)\}$ does not form a loop, because three cell entires (2, 3), (2, 4) and (2, 5) lie in the same row (second).



(i) Loop L (ii) Non-loop L'

Fig. 6.1

Theorem 6.5. *Every loop has an even number of cells.*

Proof. For any loop, we can always choose arbitrarily a starting point and a direction by an arrow mark ($\rightarrow$). We consider a loop formed by n number of cells which are consecutively numbered from 1 to n. Now assume that cell 1 and 2 exist in the same column. Thus the step from cell 1 to cell 2 involves a row change. Obviously, step from cell 2 to cell 3 must involve a column change, from cell 3 to cell 4 a row change, and so on; in general, the step to cell k involves a row change, if and only if, k is even. Since the step to cell 2 involved a row change, the step from cell n to cell 1 must be a column change and the step from cell $n-1$ to cell n a row change. Hence n will be even.

Def. (Set Containing a Loop.) *A set X of cells of a transportation table is said to* **contain a loop** *if the cells of X or of a subset of X can be sequenced (ordered) so as to form a loop.*

Theorem 6.6 (Linear Dependence and Loops). *Let X be a set of column vectors of the coefficient matrix of a T.P.. Then, a necessary and sufficient condition for vectors in X to be linearly dependent is that the set of their corresponding cells in the transportation table contains a loop.*

Proof. Let us consider an m-origin, n-destination T.P. expressed in its matrix form :

Minimize $z = \mathbf{CX}$; $\mathbf{C}, \mathbf{X} \in \mathbf{R}^{mn}$, subject to the constraints : $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, $\mathbf{b} \in \mathbf{R}^{m+n}$

where $\mathbf{b} = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$, $\mathbf{A}$ is an $(m+n) \times mn$ real matrix containing the coefficients of constraints and $\mathbf{C}$ is the cost vector.

To prove, the condition is sufficient :

Let us assume that the cells associated with the vectors of $\mathbf{X}$ contain a loop

$$L = \{ (i, j), (i, k), (l, k), \ldots, (p, o), (p, j) \}$$

If $\mathbf{a}_{ij}$ denotes the column vector of matrix $\mathbf{A}$ associated with the variable x_{ij} [the cell (i, j)], then it follows from the discussion in **sec. 10·3** that $\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$, where $\mathbf{e}_i, \mathbf{e}_{m+j} \in \mathbf{R}^{m+n}$ are unit vectors. Thus $\mathbf{X}$ includes the column vectors :

$$\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j},\ \mathbf{a}_{ik} = \mathbf{e}_i + \mathbf{e}_{m+k},\ \mathbf{a}_{lk} = \mathbf{e}_l + \mathbf{e}_{m+k},\ \mathbf{a}_{lm} = \mathbf{e}_l + \mathbf{e}_{m+m}, \ldots, \mathbf{a}_{po} = \mathbf{e}_p + \mathbf{e}_{m+o},$$

and $$a_{pj} = e_p + e_{m+j}.$$

Hence by successive addition and subtraction, we get $\mathbf{a}_{ij} - \mathbf{a}_{ik} + \mathbf{a}_{lk} - \mathbf{a}_{lm} + \ldots + \mathbf{a}_{po} - \mathbf{a}_{pj} = \mathbf{0}$

(by *Theorem 6.5*, a loop contains an even number of cells)

Therefore, this particular subset of $\mathbf{X}$, and hence $\mathbf{X}$ itself, is a *linearly dependent* set.

To prove, the condition is necessary :

Let us assume that $\mathbf{X}$ is a linearly dependent set. Then, there must exist scalars λ_{ij} not all zero such that

$$\Sigma \lambda_{ij} \mathbf{a}_{ij} = 0, \text{ where } \mathbf{a}_{ij} \in \mathbf{X}.$$

For simplification, remove all those vectors from $\mathbf{X}$ for which $\lambda_{ij} = 0$.

Now we choose arbitrarily a vector from the remaining vectors in $\mathbf{X}$. Let it be $\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$. We claim that $\mathbf{X}$ must contain at least one more vector whose second subscript is j. Suppose to the contrary that it does not, then since $\lambda_{ij} \neq 0$, the $(m+j)$th component of the vector equation $\Sigma \lambda_{ij} \mathbf{a}_{ij} = \mathbf{0}$ gives $\lambda_{ij}.1 = 0$ implying $\lambda_{ij} = 0$, a contradiction. So $\mathbf{X}$ contains at least one more vector with second subscript j.

Suppose that this vector is $\mathbf{a}_{kj} = \mathbf{e}_k + \mathbf{e}_{m+j}$. By similar reasoning, we conclude that there must be at least one more vector in $\mathbf{X}$ with the first subscript k; say, $\mathbf{a}_{kl} = \mathbf{e}_k + \mathbf{e}_{m+l}$. By same argument once again, $\mathbf{X}$ must contain at least one vector with the second subscript l. Let it be, say, $\mathbf{a}_{il} = \mathbf{e}_i + \mathbf{e}_{m+l}$.

Thus we have determined four vectors in $\mathbf{X}$, namely $\mathbf{a}_{ij}$, $\mathbf{a}_{kj}$, $\mathbf{a}_{kl}$ and $\mathbf{a}_{il}$ whose corresponding cells, form a loop. Thus the proof is complete.

If the last vector is $\mathbf{a}_{nl} = \mathbf{e}_n + \mathbf{e}_{m+l}$ instead of $\mathbf{a}_{il}$, then as explained just before there must exist at least one more vector with first subscript n. If it is $\mathbf{a}_{nj}$, a loop is complete, if not, let it be $\mathbf{a}_{n0} = \mathbf{e}_n + \mathbf{e}_{m+0}$. $\mathbf{X}$ must contain at least one more vector with second subscript 0. Now two cases will arise :

(1) The first subscript of newly discovered vector is one that has already been identified. In this case a loop has been completed.

(2) The first subscript of the newly discovered vector is also new. In this case, since the number of vectors in $\mathbf{X}$ is finite (by extending the above reasoning), we conclude that eventually a loop must be formed.

Corollary. *A feasible solution to a transportation problem is basic if, and only if, the corresponding cells in the transportation table do not contain a loop.*

Proof : Left as an exercise.

This corollary provides us a method to verify whether the current feasible solution to the transportation problem is basic or not.

Q. 1. A feasible solution to a transportation problem is basic, if and only if, the corresponding cells in the transportation table do not contain..................

2. With reference to a transportation problem define the following terms :
(i) Feasible solution (ii) Basic feasible solution, (iii) Optimal solution, (iv) Non-degenerate basic feasible solution.

3. Define 'loop' in a transportation table. What role do they play ? [Madurai B.Sc (Math.) 94]

4. In the classical transportation problem explain as to how many independent equations are there when there are *m*-origins and *n*-destinations. What happens and how to handle the solution, when the initial assignment in the problem gives less than this number of occupied cells ?

Initial Basic Feasible Solution

6.8 THE INITIAL BASIC FEASIBLE SOLUTION TO TRANSPORTATION PROBLEM

Methods of finding an *optimal solution* of the transportation problem will consist of two main steps :

(i) ***To find an initial basic feasible solution :***

(ii) ***To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible.***

There will be fewer improvements to make if initially we start with a better initial basic feasible solution. So first we shall discuss below the methods for obtaining initial basic feasible solution of a T.P.

Remark : Although the transportation problem can be solved using the regular simplex method, its special properties provide a more convenient method for solving this type of problems. This method is based on the same theory of simplex method. It makes use, however, of some shortcuts which provide a less burdensome computational scheme.

6·8-1 Methods for Initial Basic Feasible Solution

Some simple methods are described here to obtain the initial basic feasible solution of the transportation problem. These methods can be easily explained by considering the following numerical example. However, the relative efficiency of these methods is still unanswerable.

Example 1. *Find the initial basic feasible solution of the following transportation problem.*

Table 6.3

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution.

First Method : ***North-West Corner Rule (Steping Stone Method)***

[JNTU (BE Comp. Sc.) 2004; IAS (Maths. 96]

In this method, first construct an empty 3 × 4 matrix complete with row and column requirements (***Table 6·4***).

Table 6·4

	W_1	W_2	W_3	W_4	Available
F_1					7
F_2					9
F_3					18
Requirements→	5	8	7	14	

Insert a set of allocations in the cells in such a way that the total in each row and each column is the same as shown against the respective rows and columns. Start with cell (1, 1) at the north-west corner (upper left-hand corner) and allocate as much as possible there. In other words, $x_{11} = 5$, the maximum which can be alloocated to this cell as the total requirement of this column is 5. This allocation ($x_{11} = 5$) leaves the surplus amount of 2 units for row 1 *(Factory F_1)*, so allocate $x_{12} = 2$ to cell (1, 2). Now, allocations for first row and first column are complete, but there is a deficiency of 6 units in column 2. Therefore, allocate $x_{22} = 6$ in the cell (2, 2). Column 1 and column 2 requirements are satisfied, leaving a surplus amount of 3 units for row 2. So allocate $x_{23} = 3$ in the cell (2, 3), and column 3 still requires 4 units. Therefore, continuing in this way, from left to right and top to bottom, eventually complete all requirements by an allocation $x_{34} = 14$ in the *south-east* corner. Table 10·5 shows the resulting feasible solution.

Table 6·5

5 (19)	2 (30)			7
	6 (30)	3 (40)		9
		4 (70)	14 (20)	18
5	8	7	14	

On multiplying each individual allocation by its corresponding unit cost in '( )' and adding, the total cost becomes = 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015 .

Q. Explain the application of North-West Corner Rule with an example.

Second Method : *The Row Minima Method*

Step 1. The transportation table of the given problem has 12 cells. Following the ***row minima method,*** since min (19, 30, 50, 10) = 10 , the first allocation is made in the cell (1, 4), the amount of the allocation is given by x_{14} = min (7 , 14) = 7 . This exhausts the availability from factory F_1 and thus we cross-out the first row from the transportation table (Table 6.6).

Table 6·6

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	x
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	5	8	7	7	

Table 6·7

	W_1	W_2	W_3	W_4	
F_1				7	x
F_2	(70)	8 (30)	(40)	(60)	9
F_3	(40)		(70)	(20)	18
	5	8	7	7	

Step 2. In the resulting transportation table (**Table 6·7**) , since min (70, 30, 40, 60) = 30 , the second allocation is made in the cell (2, 2), the amount of allocation being x_{22} = min (9, 8) = 8 . This satisfies the requirement of warehouse W_2 and thus we cross-out the second column from the transportation table obtaining new **Table 6·8 .**

Step 3. In *Table* 13·8 , since min (70, 40, 60) = 40 ,the third allocation is made in the cell (2, 3), the amount being x_{23} = min [1, 7] = 1 . This exhausts the availability from factory F_2 ,

Table 6·8

	W_1	W_2	W_3	W_4	
F_1				7	x
F_2	(70)	8 (30)	1 (40)	(60)	x
F_3	(40)		(70)	(20)	18
	5	x	6	7	

Table 6·9

	W_1	W_2	W_3	W_4	
F_1				7	x
F_2	(70)	8 (30)	1 (40)	(60)	x
F_3	(40)		(70)	(20)	18
	5	x	6	7	

and thus we cross-out the second row from the **Table 6.8** getting the **Table 6·9 ,**

Step 4. The next allocation is made in the cell (3,4), since min (40, 70, 20) = 20, the amount of allocation being x_{34} = min (7, 18) = 7. This exhausts the requirement of warehouse W_4 and thus we cross-out the fourth column from the ***Table 6·9***.

Step 5. The next allocation is made in the cell (3, 1), since min (40, 70) = 40, the amount of allocation being x_{31} = min (5, 11) = 5. This satisfies the requirement of warehouse W_1 and so we cross-out the first column W_1 to get new ***Table 6.11.***

Table 6·10

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2		8 •	1 •		x
F_3	(40)		(70)	7 •	11
	5	x	6	x	

Table 6·11

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2		8 •	1 •		x
F_3	5 •		6 • (70)	7 •	x
	x	x	x	x	

Step 6. The last allocation of amount x_{33} = 6 is obviously made in the cell (3, 3). This exhausts the availability from factory F_3 and requirement of warehouse W_3 simultaneously. So we cross-out third row and third column to get the final solution ***Table 6·12***.

Since the basic cells indicated by (•) do not form a loop, an initial basic feasible solution has been obtained. The solution is displayed in ***Table 6·12***.

Table 6·12

	W_1	W_2	W_3	W_4	
F_1				7 •	×
F_2		8 •	1 •		×
F_3	5 •		6 •	7 •	×
	×	×	×	×	

The transportation cost is given by
$z = 7 \times 10 + 8 \times 30 + 1 \times 40 + 5 \times 40 + 6 \times 70 + 7 \times 20$
= Rs. 1110.

Third Method : *The Column Minima Method.*

This method is similar to **row-minima method** except that we apply the concept of minimum cost on columns instead of rows. So, the reader can easily solve the above problem by column minima method also.

Fourth Method : *Lowest Cost Entry Method* (Matrix Minima Method).

The initial basic feasible solution obtained by this method usually gives a lower beginning cost. In this method, first write the cost and requirements matrix (***Table 6·13***).

Start with the lowest cost entry (8) in the cell (3, 2) and allocate as much as possible, *i.e.*, $x_{32} = 8$. The next lowest cost (10) lies in the cell (1, 4), so allocate $x_{14} = 7$. The next lowest cost (19) lies in the cell (1, 1), so make no allocation, because the amount available from factory F_1 was already used in the cell (1, 4). Next lowest cost entry is (20) in the cell (3, 4) where at the most it is possible to allocate $x_{34} = 7$ in order to complete the requirements of 7 units in column 4.

Further, next lowest cost is (30) in cells (2, 2) and (1, 2) so no allocation is possible, because the requirement of column 2 has already been exhausted. This way, required feasible solution is obtained (*Table 6·13*).

This feasible solution results in lower transportation cost, *i.e.*,
2(70) + 3(40) + 8(8) + 8(40) + 7(10) + 7(20) = Rs. 814.
This cost is less by Rs. 201, *i.e.*, Rs. (1015–814) as compared to the cost obtained by *north-west corner rule*.

Table 6·13

				Available
•(19)	•(30)	•(50)	7(10)	7
2(70)	•(30)	7(40)	•(60)	9
3(40)	8(8)	•(70)	7(20)	18
Requirements 5	8	7	14	

Q. Explain the application of Matrix-Minimum method with an example.

Fifth Method : *Vogel's Approximation Method (Unit Cost Penalty Method)*

Step 1. In lowest cost entry method, it is not possible to make an allocation to the cell (1, 1) which has the second lowest cost in the matrix. It is trivial that allocation should be made in at least one cell of each row and each column.

Table 6·14

	W_1	W_2	W_3	W_4	Available
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(8)	(70)	(20)	18
Requirement	5	8	7	14	

Step 2. Next enter the *difference between the lowest and second lowest cost* entries in each column beneath the corresponding column, and put the defference between the lowest and second lowest cost entries of each row to the right of that row. Such individual differences can be thought of a **penalty** for making allocations in second lowest cost entries instead of lowest cost entries in each row or column. For example, allocate one unit in the second lowest cost cell (3, 1) instead of cell (1, 1) with lowest unit cost (19). There will be a loss (penalty) of Rs 21 per unit. In case, the lowest and second lowest costs in a row/ column are equal, the penalty will be taken zero.

Table 6·15

	W_1	W_2	W_3	W_4	Available	Penalties
F_1	*(19)	*(30)	*(50)	*(10)	7	(9)
F_2	*(70)	*(30)	*(40)	*(60)	9	(10)
F_3	*(40)	8 (8)	*(70)	*(20)	18/10	(12)
Requirements :	5	8/0	7	14		
Penalties :	(21)	(22) ↑	(10)	(10)		

Step 3. Select the row or column for which the ***penalty*** is the largest, *i.e.*, (22) (***Table*** **6·15**), and allocate the maximum possible amount to the cell (3, 2) with the lowest cost (8) in the particular column (row) making $x_{32} = 8$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

Step 4. Cross-out that column (row) in which the requirement has been satisfied. In this example, second column has been crossed-out. Then find the corresponding penalties correcting the amount available from factory F_3. Construct the first reduced penalty matrix ***Table* 6·16.**

Table 6·16

	W_1	W_3	W_4	Available	Penalties
F_1	5(19)	*(50)	*(10)	7	(9)
F_2	*(70)	*(40)	*(60)	9	(20)
F_3	*(40)	*(70)	*(20)	10 (Note)	(20)
Requirements	5/0	7	14		
Penalties	(21) ↑	(10)	(10)		

Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column 1 giving the second reduced matrix (***Table* 6·17**).

Table 6·17

	W_3	W_4	Available	Penalty
F_1	*(50)	*(10)	2 (Note)	(40)
F_2	*(40)	*(60)	9	(20)
F_3	*(70)	10(20)	10/0	(50) ←
Requirements :	7	14/4		
Penalties :	(10)	(10)		

The largest penalty (50) is now associated with the cell (3, 4) therefore allocate $x_{34} = 10$. Eliminating the row 3, the third reduced penalty matrix ***Table* 6·18** is obtained.

Table 6·18

	W_3	W_4	Available	Penalties
F_1	*(50)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirements :	7	4/2/0 (Note)		
Penalties :	(10)	(50) ↑		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct ***Table* 6·19** for the required feasible solution.

The total cost is :
$5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) =$ Rs. 779.
This cost is Rs. 35 less as compared to the cost obtained by ***Lowest Cost Entry Method.***

Table 6·19

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Requirements :	5	8	7	14	

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, but it reduces the time in reaching the optimal solution.

Short-cut. After a little practice, students may prefer to perform the entire procedure of Vogel's method within the original cost requirement ***Table 6·14***. It needs merely to cross-out rows and columns as and when they are completed and to revise requirements, available supplies and penalties as shown below.

	W_1	W_2	W_3	W_4	Available	Penalties
F_1	5 • (19)	(30)	(50)	2 • (10)	7/2/0	(9/9/40/40)
F_2	(70)	(30)	7 • (40)	2 • (60)	9/2/0	(10/20/20/20)
F_3	(40)	8 • (8)	(70)	10 • (20)	18/10/0	(12/20/50)
Required	5/0	8/0	7/0	14/4/2/0		
Penalties	(21/21)	(22) ↑	(10/10/10/10)	(10/10/10/50)		

Example 2. *Obtain an initial basic feasible solution to the following transportation problem:*

		Stores				
		I	II	III	IV	Availability
	A	7	3	5	5	34
Warehouses	B	5	5	7	6	15
	C	8	6	6	5	12
	D	6	1	6	4	19
Demand		21	25	17	17	80

[M.G. Univ., (M. Com.) 98]

[**Ans.** By Vogel's (Penalty) Method, the initial solution is:
$x_1 = 6$, $x_{12} = 6$, $x_{13} = 7$, $x_{14} = 5$, $x_{21} = 15$, $x_{34} = 12$, $x_{42} = 19$; Total transportation cost = Rs. 324.]

Q. Explain the use of Vogel's Approximation Method (VAM) with an example.

6·8-2 Summary of Methods for Initial BFS

The methods for obtaining an initial basic feasible solution to a transportation problem can be summarized as follows:

I. North-West Corner Rule (Stepping Stone Method)

Step 1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1, 1) of the transportation table.

Step 2. *(i)* If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

(ii) If $b_1 < a_1$, move horizontally right-side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

(iii) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1) = 0$ in the cell (2,1).

Step 3. Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

II. The Row Minima Method

Step 1. The smallest cost in the first row of the transportation table is determined. Let it be c_{ij}. Allocate as much as possible amount $x_{1j} = \min(a_1, b_j)$ in the cell $(1, j)$, so that either the capacity of origin O_1 is exhausted, or the requirement at destination D_j is satisfied or both.

Step 2. *(i)* If $x_{1j} = a_1$ so that the availability at origin O_1 is completely exhausted, cross-out* the first row of the table and move down to the second row.

(ii) If $x_{1j} = b_j$ so that the requirement at destination D_j is satisfied, cross-out the jth column and re-consider the first row with the remaining availability of origin O_1.

(iii) If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is completely satisfied. An arbitrary tie-breaking choice is made. Cross-out the jth column and make the second allocation $x_{1k} = 0$ in the cell $(1, k)$ with c_{1k} being the new *minimum* cost in the first row. Cross-out the first row and move down to the second row.

Step 3. Repeat *steps 1 and 2* for the reduced transportation table until all the requirements are satisfied.

III. The Column Minima Method

Step 1. Determine the smallest cost in the first column of the transportation table. Let it be c_{i1}. Allocate $x_{i1} = \min(a_i, b_1)$ in the cell $(i, 1)$.

Step 2. *(i)* If $x_{i1} = b_1$, cross-out the first column of the transportation table and move towards right to the second column.

* By saying *"crossout a row or a column"* we shall mean that no cell from that row or column can be chosen for the basis entry at a later step.

(ii) If $x_{i1} = a_i$, cross-out the ith row of the transportation table and reconsider the first column with the remaining demand.

(iii) If $x_{i1} = b_j = a_i$, cross-out the ith row and make the second allocation $x_{k1} = 0$ in the cell $(k, 1)$ with c_{k1} being the new minimum cost in the first column. Cross-out the column and move towards right to the second column.

Step 3. Repeat *steps 1 and 2* for the reduced transportation table until all the requirements are satisfied.

IV. Lowest Cost Entry Method (LCEM) or Matrix Minima Method

Step 1. Determine the smallest cost in the cost matrix of the transportation table. Let it be (c_{ij}). Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j).

Step 2. *(i)* If $x_{ij} = a_i$, cross-out the ith row of the transportation table and decrease b_j by a_i. Go to *step* 3.

(ii) If $x_{ij} = b_j$, cross out the jth column of the transportation table and decrease a_i by b_j. Go to *step* 3.

(iii) If $x_{ij} = a_i = b_j$, cross-out either the ith row or jth column but not both.

Step 3. Repeat *steps 1 and 2* for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

V. Vogel's Approximation Method (VAM)

Step 1. For each row of the transportation table identify the *smallest* and *next-to-smallest* cost. Determine the difference between them for each row. These are called *'penalties'*. Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.

Step 2. Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty correspond to ith row and let c_{ij} be the smallest cost in the ith row. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross-out the ith row or the jth column in the usual manner.

Step 3. Again compute the column and row penalties for the reduced transportation table and then go to *step* 2. Repeat the procedure until all the requirements are satisfied.

Q. 1. Explain with an example the North-West corner rule, the least cost method, and the Vogel's Approximation method for obtaining an initial basic feasible solution of a transportation problem. **[C.A. (Nov.) 91]**

2. Explain Vogel's Approximation Method of solving a transportation problem.

3. Explain the lowest cost entry method for obtaining an initial basic solution of a transportation problem. **[Madurai B.Sc. (Comp. Sc.) 92]**

4. List the various methods that can be used for obtaining an initial basic feasible solution for a transportation problem and describe any one of them. **[Garhwal M.Sc. (Math.) 95; Delhi B.Sc (Math.) 93]**

5. Discuss the algorithm of stepping stone method. **[VTU (BE Mech.) 2002]**

EXAMINATION PROBLEMS

1. Determine an initial basic feasible solution to the following transportation problem using the north-west corner rule, where O_i and D_j represent ith origin and jth destination respectively.

(i)

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

[IAS (Maths.) 89]

(ii)

		I	II	III	IV	Supply
	A	13	11	15	20	2,000
From	B	17	14	12	13	6,000
	C	18	18	15	12	7,000
Demand		3,000	3,000	4,000	5,000	

[Bharathidasan B.Sc (Math.) 90]

[**Ans.** (i) $x_{11} = 6$, $x_{12} = 8$, $x_{22} = 2$, $x_{23} = 14$, $x_{33} = 1$, $x_{34} = 4$, cost = Rs. 128
(ii) $x_{11} = 2$, $x_{21} = 1$, $x_{22} = 3$, $x_{23} = 2$, $x_{33} = 2$, $x_{34} = 5$]

Note. In 1989, a new method for initial solution of transportation problem was developed by the Author of this book. The brief outlines of this new method are given in the Appendix. This method provides the initial solution very near to optimal solution. In most of the cases, the initial solution obtained by this new method proves to be optimum.

(iii) To

							Available
From	9	12	9	8	4	3	5
	7	3	6	8	9	4	8
	4	5	6	8	10	14	6
	7	3	5	7	10	9	7
	2	3	8	10	2	4	3
Require	3	4	5	7	6	4	

[Ans. $x_{11} = 3$, $x_{12} = 2$, $x_{22} = 2$, $x_{23} = 5$, $x_{24} = 1$, $x_{34} = 6$, $x_{44} = 0$, $x_{45} = 6$, $x_{46} = 1$, $x_{56} = 3$.]

2. Determine an initial basic feasible solution to the following T.P. using the row/column minima method.

(i)

From	To			Availability
	A	B	C	
I	50	30	220	1
II	90	45	170	4
III	250	200	50	4
Require	4	2	3	

[Bharathidasan B.Sc (Math.) 90]

(ii) To

	A	B	C	D	Available
From I	6	3	5	4	22
From II	5	9	2	7	15
From III	5	7	8	6	8
Demand	7	12	17	9	

[Ans. (i) $x_{12} = 1$, $x_{21} = 3$, $x_{22} = 1$, $x_{31} = 1$, $x_{33} = 3$ (ii) $x_{12} = 12$, $x_{13} = 1$, $x_{14} = 9$, $x_{23} = 15$, $x_{31} = 7$, $x_{33} = 1$]

3. Obtain an initial basic feasible solution to the following T.P. using he matrix minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

where O_i and D_j denote ith origin and jth destination respectively.

[Ans. $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = 0$, $x_{33} = 6$]

4. Find the initial basic feasible solution of the transportation problem where cost-matrix is given below :

Destination

		A	B	C	D	Supply
Origin	I	1	5	3	3	34
	II	3	3	1	2	15
	III	0	2	2	3	12
	IV	2	7	2	4	19
Demand		21	25	17	17	

[Ans. $x_{11} = 9$, $x_{12} = 8$, $x_{14} = 17$, $x_{23} = 15$, $x_{31} = 12$, $x_{42} = 17$, $x_{43} = 2$; cost = Rs. 238, using *'lowest cost entry method'*].

5. Determine an initial basic feasible solution using (i) Vogel's method and (ii) Row minima method, by considering the following transportation problem :

Destination

		1	2	3	4	Supply
Source	1	21	16	15	13	11
	2	17	18	14	23	13
	3	32	27	18	41	19
Demand		6	10	12	15	43

[VTU (BE Mech.) 2002; Gauhati (MCA) 92]

[Ans. $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$, cost = Rs. 686]

6. Determine an initial basic feasible solution to the following T.P. using : (*a*) North-west corner rule, and (*b*) Vogel's method.

		Destination					
		A_1	B_1	C_1	D_1	E_1	Supply
	A	2	11	10	3	7	4
Origin	B	1	4	7	2	1	8
	C	3	9	4	8	12	9
Demand		3	3	4	5	6	21

[JNTU (B. Tech.) 2000]

[Ans. *North west corner Rule*: $x_{11}=3$, $x_{12}=1$, $x_{22}=2$, $x_{23}=4$, $x_{24}=2$, $x_{34}=3$, $x_{35}=6$, cost = Rs. 153 .]
***Vogel's Method*:** $x_{14}=4$, $x_{22}=2$, $x_{35}=6$, $x_{34}=3$, $x_{32}=1$, $x_{33}=4$, $x_{34}=1$, cost = Rs. 68].

7. Use *north west corner rule* to determine an initial basic feasible solution to the following T.P. when does it have a unique solution ?

		To			
		A	B	C	Supply
	a	2	7	4	5
From	b	3	3	1	8
	c	5	4	7	7
	d	1	6	2	14
Demand		7	9	18	34

[Meerut B.Sc. (Math.) 90]

Does the use of matrix minima method give a better (improved) basic feasible solution ? Why ?
[Ans. $x_{11}=5$, $x_{21}=2$, $x_{22}=6$, $x_{32}=3$, $x_{33}=4$, $x_{43}=14$, cost = Rs. 102; Yes]

8. Determine an initial basic feasible solution to the following T.P. using : (*a*) matrix minima method, (*b*) Vogel's approx. method.

	Destination				
	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Demand	20	40	30	10	100

[Meerut 2002; IAS (Maths) 88]

[Ans. For (*a*) and (*b*) bold : $x_{11}=20$, $x_{13}=10$, $x_{22}=20$, $x_{23}=20$, $x_{23}=20$, $x_{24}=10$, $x_{32}=20$, cost = Rs. 180]

9. (i) Explain vogel's method by obtaining initial BFS of the following transportation problem :

	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	

[Ans. $x_{11}=9$, $x_{12}=3$, $x_{21}=5$, $x_{23}=7$, $x_{33}=16$.]

		To			
		I	II	III	Supply
	A	50	30	200	1
From	B	90	45	170	3
	C	250	200	50	4
Demand		4	2	2	

[Bharathidasan B.Sc (Math.) 90]

[Ans. $x_{11}=1$, $x_{21}=3$, $x_{31}=0$, $x_{32}=2$ $x_{33}=2$]

10. Determine an initial feasible solution to the following T.P. using (a) (North-West corner rule, and (b) Vogel's approximation method :

Origin	Destination				Supply
	D_1	D_2	D_3	D_4	
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	950

[Ans. $x_{11}=200$, $x_{12}=50$, $x_{22}=175$, $x_{24}=125$, $x_{33}=275$, $x_{34}=125$.]

Optimum Solution

6.9 MOVING TOWARDS OPTIMALITY

After obtaining an initial basic feasible solution to a given transportation problem, the next question is 'how to arrive at the optimum solution'. The basic steps for reaching the optimum solution are the same as given for simplex method, namely :

Step 1. Examination of initial basic feasible solution for *non-degeneracy.* If it is *degenerate,* some modification is required to make it non-degenerate (as discussed in *Sec.* **6.11**).

Step 2. *(i)* Determination of net-evaluations (cost-difference) for empty cells.
(ii) Optimality test of current solution.

Step 3. Selection of the entering variable, provided *Step 2(ii)* indicates that the current solution can be improved.

Step 4. Selection of the leaving variable.

Step 5. Finally, repeating the ***steps 1 through 4*** until an optimum solution is obtained.

6·9-1 To Examine the Initial Basic Feasible Solution for Non-degeneracy

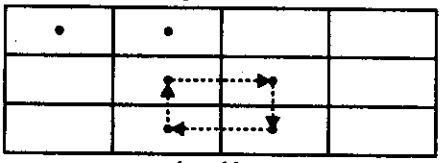
A basic feasible solution of an $m \times n$ transportation problem is said to be *non-degenerate,* if it has the following two properties :

(1) *Initial BFS must contain exactly $m + n - 1$ number of individual allocations.*
For example, in 3×4 transportation problem, the number of individual allocations in BFS obtained by any one of the methods discussed so far is equal to 6, *i.e.*, $3 + 4 - 1$, which can be easily verified from ***Tables* 6·5, 6·12, 6·13 and 6·19.**

(2) *These allocations must be in 'independent positions.'*
Independent positions of a set of allocations mean that it is always impossible to form any closed loop through these allocations. ***Tables* 6·20, 6·21** show the non-independent, and ***Table* 6·23** independent positions by '•'.

Table 6·20

Non-Independent Positions



closed loop

Table 6·21

Non-Independent Positions

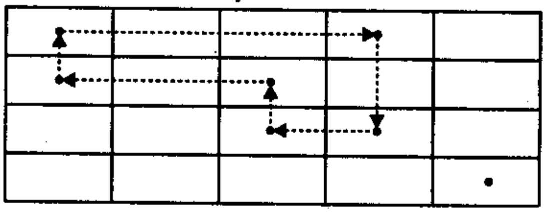


Table 6·22

Non-Independent Positions

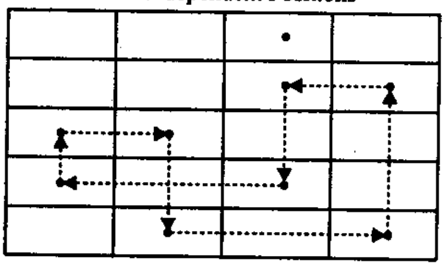


Table 6·23

Independent Positions

•				•
			•	•
		•		•

In above allocation patterns of different problems, the dotted lines constitute what are known as ***loops***. A loop may or may not involve all allocations. It consists of (at least 4) horizontal and vertical lines with an allocation at each corner which, in turn, is a join of a horizontal and vertical lines. At this stage, loop of ***Table 6·22*** *should be particularly noted. Here two lines intersect each other at cell (4, 2) and do not simply join ; therefore this is not to be regarded as a corner. Such allocations in which a loop can be formed are known as non-independent positions whereas those (of* ***Table 6·23***) in which a loop cannot be formed are regarded as independent.

6·9-2 Determination of Net-Evaluations (u, v method)

Unlike the simplex method, the net-evaluations for a transportation problem can be determined more easily by using the properties of the *primal* and *dual* problems.

Let us consider the following m-origin, n-destination transportation problem :

Determine x_{ij} so as to minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} (c_{ij})$, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{or} \quad a_i - \sum_{j=1}^{n} x_{ij} = 0, \qquad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{or} \quad b_j - \sum_{i=1}^{m} x_{ij} = 0, \qquad \text{for } j = 1, 2, \ldots, n$$

and $\quad x_{ij} \geq 0$, for all i and j .

Let $u_1, u_2, \ldots u_m$ and $v_1, v_2, \ldots v_n$ be the dual variables associated with the above *origin* and *destination* constraints, respectively. Since the above *primal* T.P. has $m + n$ constraint equations in mn number of variables, so the *dual* of above problem will contain mn constraints in $m + n$ dual variables in the form :

$$u_i + v_j \leq c_{ij} \text{ and } u_i, v_j \text{ are unrestricted for all } i \text{ and } j \ (\because \text{ constraints in the primal are equations})$$

From ***Duality of L.P.***, we know that for any standard primal L.P.P. with basis $\mathbf{B}$ and associated cost vector $\mathbf{C_B}$, the associated solution to its dual is given by $\mathbf{W_B} = \mathbf{C_B} \mathbf{B}^{-1}$. Thus, if $\mathbf{a}_j$ is the jth column of the primal constraint matrix, then an expression for evaluating the net-evaluation for minimization problem is given by

$$c_j - z_j = c_j - \mathbf{C_B} (\mathbf{B}^{-1} \mathbf{a}_j) = c_j - \mathbf{W_B a_j} \quad \text{for all } j .$$

But, in the present case of *rectangular* transportation problem (which is the special case of L.P.P.), the dual solution can be represented by

$$(\mathbf{u}, \mathbf{v}) = (u_1, \ldots, u_m, v_1, \ldots, v_n)$$

and therefore the net evaluations are analogously obtained by simply replacing $c_j \to c_{ij}$, $\mathbf{W_B} \to (\mathbf{u}, \mathbf{v})$, $\mathbf{a}_j \to \mathbf{a}_{ij}$ in the above formula to get

$$c_{ij} - z_{ij} = c_{ij} - (\mathbf{u}, \mathbf{v}) \mathbf{a}_{ij} = c_{ij} - (u_1, \ldots, u_m, v_1, \ldots, v_n) [\mathbf{e}_i + \mathbf{e}_{m+j}]$$
$$= c_{ij} - (u_i + v_j) ; i = 1, \ldots, m ; j = 1, \ldots, n ,$$

where $\mathbf{a}_{ij}$ is the column vector of the constraint matrix associated with the *rectangular* variable x_{ij} . For simplicity, we shall denote the net evaluation $c_{ij} - z_{ij}$ by d_{ij} in all further discussions.

Now, since the net evaluations must vanish for the basic variables it follows that $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic cells (i, j) where u_i and v_j satisfy the relation $c_{rs} = u_r + v_s$ for all basic cells (r, s) . Except for the degeneracy case, there are $m + n - 1$ dual equations in $m + n$ dual unknowns for the $m + n - 1$ basic cells. We can arbitrarily assign the value to one of these unknowns u_r and v_s and solve uniquely for the remaining $m + n - 1$ variables. After this arbitrary assignment, say $u_1 = 0$, the rest of the values are obtained by simple addition and subtraction. Once we determine all the u_i and v_j , the net evaluations for all the non-basic cells are easily determined by the relation $d_{ij} = c_{ij} - (u_i + v_j)$.

Alternative Method to Determine Net-Evaluations.

The necessary condition for optimality can also be established in the form of the following theorem.

Theorem 6.7. *If we have a feasible solution consisting of* $m + n - 1$ *independent allocations, and if numbers* u_i *and* v_j *satisfying* $c_{rs} = u_r + v_s$ *, for each* ***occupied*** *cell* (r, s)*, then the evaluation* d_{ij} *corresponding to each* ***empty cell*** (i, j)*, is given by* $d_{ij} = c_{ij} - (u_i + v_j)$.

Proof. The transportation problem is to find $x_{ij} \geq 0$ in order to minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij}) \qquad ...(6\cdot6)$$

subject to the restrictions

$$\sum_{j=1}^{n} x_{ij} = a_i, \; i = 1, 2, 3, \ldots, m \qquad ...(6\cdot7)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \; j = 1, 2, 3, \ldots, n \qquad ...(6\cdot8)$$

and $\quad x_{ij} \geq 0$ for all i and j.

The restrictions (6·7) and (6·8) may be written as

$$0 = a_i - \sum_{j=1}^{n} x_{ij}, \; i = 1, 2, 3, \ldots, m \qquad ...(6\cdot9)$$

$$0 = b_j - \sum_{i=1}^{m} x_{ij}, \; j = 1, 2, 3, \ldots, n. \qquad ...(6\cdot10)$$

Any multiple of each of these restrictions [(6·9) and (6·10)] can be legally added to the objective function (6·6) to try to eliminate the basic variable. These multiples are denoted by u_i ($i = 1, 2, \ldots, m$) and v_j ($j = 1, 2, \ldots, n$), respectively. Thus,

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij}) + \sum_{i=1}^{m} u_i \left(a_i - \sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} v_j \left(b_j - \sum_{i=1}^{m} x_{ij}\right) \qquad ...(6\cdot11a)$$

or

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij} - (u_i + v_j)] x_{ij} + \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j. \qquad ...(6\cdot11b)$$

The necessary condition for a coefficient of zero is

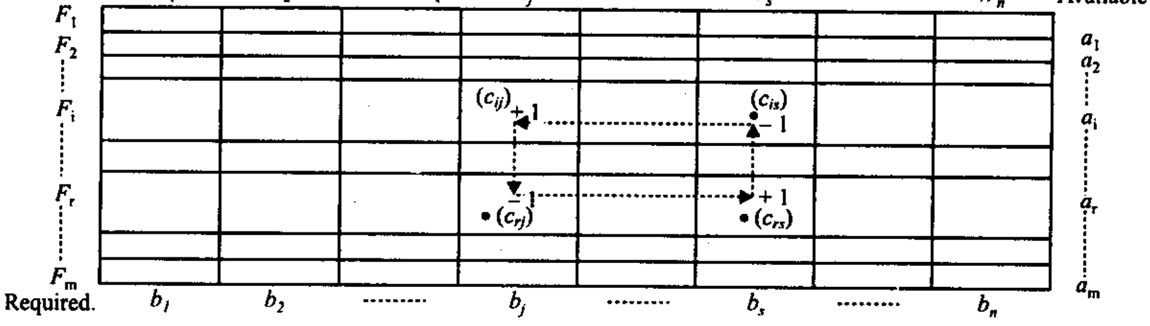
$$c_{rs} = u_r + v_s \qquad ...(6\cdot12)$$

for each basic variable x_{rs}, *i.e.*, for each occupied cell (r, s). Since there are $m + n - 1$ number of equations of the form (6·12) in $(m + n)$ number of unknowns (u_i and v_j), so if assignment is made to an arbitrary value of one of the u_i or v_j, then rest of the $(m + n - 1)$ unknowns can be easily solved algebraically. One reasonable and convenient rule, which will be adopted here, is to select the u_i which has the largest number of allocations in its row, and assign it the value of zero. And $c_{ij} = u_i + v_j$ immediately yields v_j for columns containing those allocations.

To prove the required result, first suppose that the empty cell (i, j) be connected to occupied cells by a closed loop (*Table 6·24*).

First, allocate + 1 unit to the empty cell (i, j), and in order to balance the total requirement of warehouse W_j, add − 1 unit to occupied cell (r, j). Consequently, the total amount available from factory F_r will be balanced by adding + 1 unit to occupied cell (r, s), which in turn causes column W_s to become unbalanced. So balance the column W_s by adding − 1 unit to the occupied cell (i, s).

Table 6·24



This process will give the *cost difference* d_{ij} [called the empty *cell evaluation* for (i, j)] between the new solution and the original solution.

Thus, $$d_{ij} = c_{ij} - c_{rj} + c_{rs} - c_{is} \,. \qquad ...(6\cdot13)$$

Using the result (6·12) for all occupied cells such as (r, j), (r, s) and (i, s),

$$d_{ij} = c_{ij} - (u_r + v_j) + (u_r + v_s) - (u_i + v_s) = c_{ij} - (u_i + v_j) \,. \qquad ...(6\cdot14)$$

This proves the result for a loop of square shape connecting the empty cell(i, j) to occupied cells. In a similar fashion, generalize the empty cell (i, j) to occupied cells.

This completes the proof of the theorem.

6·9-3 The Optimality Test

If the cost difference $d_{ij} \geq 0$ (which implies increase in cost for each empty cell), then the BFS under test must be optimal. Otherwise, if $d_{ij} < 0$ (because negative difference implies decrease in cost) for one or more empty cells, then it would be better to reduce the cost more by allocating as much as possible to the cell with the largest negative (smallest) value of d_{ij}. This way, it is possible to improve the BFS successively for reduced cost till the optimal solution is obtained for which $d_{ij} \geq 0$ for each empty cell.

The optimality test for given BFS of the transportation problem may be summarized as follows :

1. ***Start with a basic feasible solution consisting of $m + n - 1$ allocations in independent positions.***
2. ***Determine a set of $m + n$ numbers u_i ($i = 1, 2, 3, \ldots, m$) and v_j ($j = 1, 2, 3, \ldots, n$) such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$.***
3. ***Calculate cell evaluations (unit cost differences) d_{ij} for each empty cell (i, j) by using the formula***
$$d_{ij} = c_{ij} - (u_i + v_j) \,.$$
4. ***Finally, examine the matrix of cell evaluations d_{ij} for negative entries and conclude that –***
   - ***(i) solution under test is optimal, if none is negative ;***
   - ***(ii) alternative optimal solutions exist, if none is negative but any is zero ;***
   - ***(iii) solution under test is not optimal, if any is negative, then further improvement is required by repeating the above process.***

We now proceed to answer the question : how to improve the current BFS if it is not optimal, *i.e.* if all $d_{ij} \leq 0$.

6·9-4 Selection of Entering Variable

Here our aim is to minimize the cost of transportation. So the current basic feasible solution will not be optimum so long as any of the net evaluation d_{ij} is negative. Thus if all d_{ij} are non-negative, the current solution is an optimum one, otherwise using simplex like criterion we select such variable x_{rs} to enter the basis for which the net evaluaton $d_{rs} = \min_{i,j} \{d_{ij} < 0\}$.

6·9-5 Selection of Leaving Variable

Our next step will be to determine the leaving basic variable and then to determine the new improved basic solution. The simplex like leaving criterion in the notations of transportation problem states that if the variable x_{rs} is selected to enter the basis, then the basic variable x_{Bi} corresponding to the *minimum ratio* : $\min \left\{ \dfrac{x_{Bi}}{y_{irs}}, y_{irs} > 0 \right\}$, will leave the basis. However, due to special structure of the transportation problem the above criterion has been simplified to a great extent.

Theorem 6.8. *Let* $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m+n-1}\}$ *be a basis set for the column vectors of the coefficient matrix* A *of m-origin, n-destination transportation problem. In the representation of any non-basic* $\mathbf{a}_{rs}$ *as a linear combination :*

$$\mathbf{a}_{rs} = \sum_{i=1}^{m+n-1} y_{irs} \, \mathbf{b}_i \,,$$

of basis vectors, every scalar element (y_{irs}) *is either* – 1 or + 1 .

Proof. Since $\{\mathbf{a}_{rs}, \mathbf{b}_1, \ldots, \mathbf{b}_{m+n-1}\}$ is a linearly dependent set (by *Theorem 6·6*), the set of the associated cells contains a loop. So the cell (r, s) must be in a loop. The vectors $\mathbf{b}_i$'s are, of course, some column vectors $\mathbf{a}_{ij}$'s of A.

Suppose the set of associated cells contains the following loop :

$$L = \{(r, s), (r, t), (p, t), (p, q), \ldots, (u, v), (u, s)\},$$

where $\mathbf{a}_{rs}, \mathbf{a}_{rt}, \ldots, \mathbf{a}_{us}$ are the given basis vectors ($\leq m + n - 1$).

Now, since $\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$ for all i and j, and because the number of cells in a loop is always even, we have

$$\mathbf{a}_{rs} - \mathbf{a}_{rt} + \mathbf{a}_{pt} - \mathbf{a}_{pq} + \ldots + \mathbf{a}_{uv} - \mathbf{a}_{us} = \mathbf{0}$$

which yields, $\mathbf{a}_{rs} = \mathbf{a}_{rt} - \mathbf{a}_{pt} + \mathbf{a}_{pq} - \ldots - \mathbf{a}_{uv} + \mathbf{a}_{us}$.

This is the unique representations of $\mathbf{a}_{rs}$ as a linear combination of basis vectors; and hence the y_{irs} elements associated with the basis vector in the above representation are $+1, -1, \ldots, -1$ and $+1$.

This completes the proof of the theorem.

Thus if x_{rs} is the entering variable, the basic variable x_{Bt} will leave the basis if $x_{Bt} = \min \{x_{Bi}\}$, since positive y_{irs} is $+1$ for all basic variables.

6·9-6 Determination of New (Improved) Basic Feasible Solution

After the entering and leaving variables are determined, all that remains to determine is ***'the new (improved) basic solution'***. The usual transformation formulae for obtaining the new basic solution, in the simplex transportation notation, are given by

$$\hat{x}_{Bt} = \frac{x_{Bt}}{y_{trs}}, \quad y_{trs} > 0 \qquad \text{and} \qquad \hat{x}_{Bi} = x_{Bi} - \frac{x_{Bt}}{y_{trs}} y_{irs} \text{ for all } i \neq t$$

Since $y_{trs} = +1$, $y_{irs} = \pm 1$ for basic variables, and $y_{irs} = 0$ for non-basic variables, the above transformations get simplified to

$$\hat{x}_{Bt} = x_{Bt},$$

$$\hat{x}_{Bi} = x_{Bi} - y_{irs}\, x_{Bt} = x_{Bi} \pm x_{Bt} \text{ for all } x_{Bi} \in X, \, i \neq t$$

and $\quad \hat{x}_{Bi} = x_{Bi}, \quad$ for all $x_{Bi} \notin X$,

where X is the set of those basic variables whose corresponding cells are included in the loop as identified in the preceding theorem.

In practice, however, this improvement procedure is quite simple. The working-rule is outlined in the following steps :

6·9-7 Working Rule to Obtain Leaving Variable and Improved Basic Feasible Solution

Step 1. After identifying the entering variable x_{rs}, describe a loop which starts and ends at the non-basic cell (r, s) connecting only the basic cells. *Such a closed path exists and is unique for any non-degenerate basic solution.*

Step 2. The amount (say θ) to be allocated to the entering variable is interchangably *subtracted from* and *added to* the successive end points of the closed loop so that the supply and demand constraints always remain satisfied.

Step 3. Then the minimum value of θ, which will render non-negative values for all the basic variables in the new solution, is obtained. This consequently, determines the leaving variable.

For illustration, see ***Example 2*** in ***Sec 6·10-2***.

Q. 1. Describe any two methods of constructing a basic feasible solution for the transportation problem. How would you test for optimality of a given solution for the transportation problem? **[Meerut (Stat.) 92]**

2. Show that the transportation and assignment problem can be regarded as the particular cases of L.P.P. **[Meerut (Stat.) 98]**

6.10 TRANSPORTATION ALGORITHM FOR MINIMIZATION PROBLEM

The transportation algorithm for minimization problem can be summarized in the following steps.

The Algorithm :

Step 1. First, construct a transportation table entering the origin capacities a_i, the destination requirements b_j and the costs c_{ij}, as shown in **Table 6.14** (page 194)

Step 2. Find an initial basic feasible solution by Vogel's method or by any of the given methods. Enter the solution at the centre (•) of basic cells.

Step 3. For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$ and entering successively the values of u_i and v_j on the transportation table as shown in **Table 6.26.** (page 269)

Step 4. Compute the cost differences $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic cells and enter them in the upper right corners of the corresponding cells.

Step 5. Apply optimality test by examining the sign of each d_{ij} :

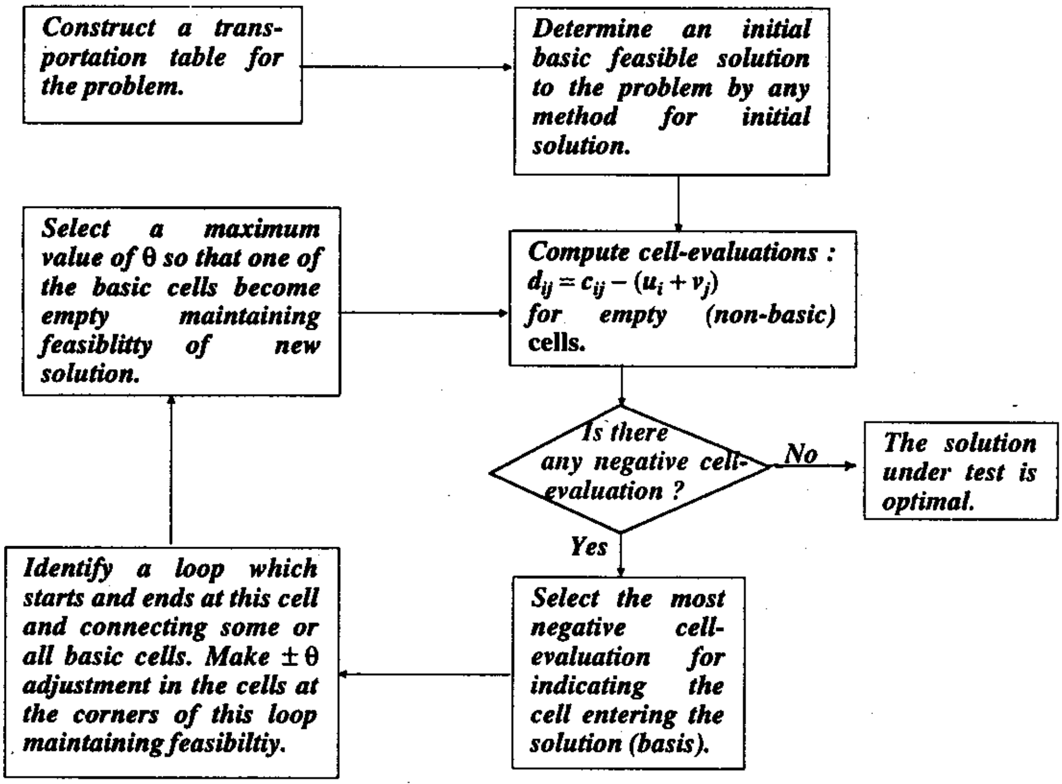
(i) If all $d_{ij} \geq 0$, the current basic feasible solution is an optimum one.

(ii) If at least one $d_{ij} < 0$ (negative), select the variable x_{rs} (having the most negative d_{rs}) to enter the basis.

Step 6. Let the variable x_{rs} enter the basis. Allocate an unknown quantity say θ, to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7. Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative (≥ 0). The basic cell whose allocation has been made zero will leave the basis.

Flowchart of Transportation Algorithim



Step 8. Now, return to *step 3* and then repeat the process until an optimum basic feasible solution is obtained.

The above iterative procedure determines an optimum solution in a finite number of steps. This method is called MODI METHOD. and can be easily remembered with the help of the FLOW-CHART (given on page 205).

Q. 1. Give an algorithm for solving transportation problem.
2. State the transportation problem. Describe clearly the steps involved in solving the problem.
3. Describe the transportation problem. Give a method of finding an initial feasible solution. Explain what is meant by an optimality test ? Give the method of improving over the initial solution to reach the optimal feasible solution. **[Meerut 94]**
4. Assume that in a transportation problem the demand and supply levels are all positive and integral. Show that there exists an integral optimal solution if the total demand equals total supply levels.
5. Describe the computational procedure of optimality test in a transportation problem.
6. Explain briefly the step-wise description of the computational procedure for solving the transportation problem. **[Delhi B.Sc. (Maths.) 91]**
7. Develop methematical model of a balanced transportation problem. Prove that it always has a feasible solution. **[IAS (Maths.) 99]**
8. How do you diagnose that the given transportation problem is having more than one optimal alternate optimal solution. **[AIMS (BE Ind.) Banglore 2002]**

6·10-1 Computational Demonstration of Optimality Test

Example 3. (a) *Obtain an initial basic feasible solution to the transportation problem of* ***Example 1****. Is this solution an optimal solution ? If not, obtain the optimal solution.*

[IGNOU 2001; JNTU (Mach.) 99; Gauhati (MCA) 91]

(b) *If a company is spending Rs. 1000 on transportation of its units to four warehouses from three factories. What can be the maximum saving by optimal scheduling.*

Solution. (a) Computational demonstration for optimality is performed by taking the initial basic feasible solution of ***Example 1*** with $m+n-1$ allocations in independent positions with transportation cost of Rs 779 obtained (by *Vogel's Method*). This initial basic feasible solution is given in **Table 6·25.**

Table 6·25

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Required	5	8	7	14	

Step 1. The initial BFS has $m+n-1$ allocations, that is, $3+4-1=6$ allocations in independent positions. Therefore, condition (1) of optimality test [in sec. *10·9-3*] is satisfied.

Step 2. Since u_i $(i=1,2,3)$ and v_j $(j=1,2,3,4)$ are to be determined by means of unit cost in the respective occupied cells only, assign a u-value of any particular amount (conveniently *zero*) to any particular row (convenient rule is to select the u_i which has the largest number of allocations in its row). Since all rows contain the same number of allocations, take any of the u_i (say u_3) equal to zero.

When $u_3=0$, $v_4=20$ (since $c_{34}=u_3+v_4$; $c_{34}=20$). Similarly, $c_{32}=u_3+v_2$ or $8=0+v_2$ or $v_2=8$. Again, $c_{14}=u_1+v_4$ or $10=u_1+20$ (since $c_{14}=10$), then $u_1=-10$. In the same way :$60=20+u_2$, which gives $u_2=40$; $19=u_1+v_1$ or $19=-10+v_1$, which gives $v_1=29$; $40=u_2+v_3$ or $40=40+v_3$, which gives $v_3=0$. This completes the set of u_i $(i=1,2,3)$ and v_j $(j=1,2,3,4)$ as shown in **Table 6·26.**

Table 6·26

					u_i
	•(19)			•(10)	-10
			•(40)	•(60)	40
		•(8)		•(20)	0
v_j	29	8	0	20	

Step 3. To compute the matrix of cell evaluations $d_{ij}=c_{ij}-(u_i+v_j)$ for *empty cells*, it is convenient to write a matrix $[c_{ij}]$ for *empty cells* and the matrix of numbers $[u_i+v_j]$ for *empty cells* only, then subtract the later matrix from the former one.

Table 6·27 *(from Table 6·3)*

Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

Table 6·28

Matrix $[u_i + v_j]$ for empty cells

•	−2	−10	•
69	48	•	•
29	•	0	•

Now, subtracting the matrix $[u_i + v_j]$ from the matrix $[c_{ij}]$, *i.e.* ***(Table 6·27 – Table 6·28),*** the following matrix $[c_{ij} - (u_i + v_j)]$ of cell evaluations is obtained.

Table 6·29 gives the empty cell evaluations : $d_{12} = 32$, $d_{13} = 60$, $d_{21} = 1$, $d_{22} = -18$, $d_{31} = 11$ and $d_{33} = 70$. The largest negative cell evaluation (marked √) is $d_{22} = -18$ which indicates that allocation of one unit to this empty cell (2, 2) will reduce the achieved cost of Rs 779 by Rs. 18. So allocate (say, θ) to cell (2, 2) as much as possible, followed by alternately subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility (non-negativity of allocations). For this purpose, the initial basic feasible solution can be read from ***Table 6·30***. It is easily seen by the following rule that at the most θ = 2 units can be allocated from cell (2, 4) to cell (2, 2) still satisfying the row and column total and non-negativity restrictions on the allocations.

Table 6·29

•	32	60	•
1	−18	•	•
11	•	70	•

Table 6.30

					Available
	5 • (19)			2 • (10)	7
		+θ √	7 • (40)	2 − θ • (60)	9
		8 − θ • (8)		10 + θ • (20)	18
Required	5	8	7	14	

A Rule to Determine θ : Reallocation is done by transfering the maximum possible amount θ in the marked (√) cell. The value of θ , in general, is obtained by equating to zero the minimum of the allocations containing − θ (not + θ) at the corners of the closed loop. That is, in *Table 6·30*, min [8 − θ , 2 − θ] = 0 or 2 − θ = 0 or θ = 2 units. Thus improved basic feasible solution is given in ***Table 6·31***.

Table 6·31

					Available
	5(19)			2(10)	7
		2(30)	7(40)		9
		6(8)		12(20)	18
Required	5	8	7	14	

The cost for this solution becomes

$$= 5\ (19) + 2\ (10) + 2\ (30) + 7\ (40) + 6\ (8) + 12\ (20) = \text{Rs } 743.$$

The cost of Rs 743 is Rs (2 × 18 = 36) less than Rs 779 which was expected also.

Step 4. Test this improved solution (*Table 6·31*) for optimality by repeating *steps 1, 2* and *3*. In each step, following matrices are obtained.

Table 6·32

Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	
(70)	•	•	(60)
(40)	•	(70)	•

Table 6·33

Matrix for u_i and v_j

					u_i
	• (19)			• (10)	−10
		• (30)	• (40)		22
		• (8)		• (20)	0
$v_j \rightarrow$	29	8	18	20	

Table 6·34

Matrix $[u_i + v_j]$ for empty cells

•	−2	8	•
51	•	•	42
29	•	18	•

Table 6·35

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	32	42	•
19	•	•	18
11	•	52	•

Since none of the cell evaluations is negative, *i.e.*, $d_{12} = 32$, $d_{13} = 42$, $d_{21} = 19$, $d_{24} = 18$, $d_{31} = 11$ and $d_{33} = 52$, the solution given in ***Table 6·31*** is optimal with minimum cost of Rs. 743.

(b) Maximum saving = Rs. 1000 − Rs. 743 = Rs. 257 . **Ans.**

6·10-2 More Solved Examples

Example 4. *Solve the following transportation problem in which cell entries represent unit costs.*

Table 6·36

	To			Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

Solution. By *Vogel's method*, the following initial basic feasible solution having the transportation cost of Rs. 80 is obtained.
To test the solution for optimality, required tables are given below.

Table 6·37

				Available
	5(2)			5
			8(1)	8
		7(4)		7
	2(1)	2(6)	10(2)	14
Required	7	9	18	

Table 6·38
[Matrix for set of u_i and v_j]

				u_i
	(2)			1
			(1)	−1
		(4)		−2
	(1)	(6)	(2)	0
v_j	1	6	2	

Table 6·39
[Matrix c_{ij} for empty cells]

•	(7)	(4)
(3)	(3)	•
(5)	•	(7)
•	•	•

Table 6·40
[Matrix $(u_i + v_j)$ for empty cells]

•	7	3
0	5	•
− 1	•	0
•	•	•

Table 6·41
Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	0	1
3	− 2 √	•
6	•	7
•	•	•

From ***Table* 6·41,** it is observed that the cell evaluation, $d_{22} = -2$, is negative. Therefore, the solution [*Table* 6·37] under test is not optimal.

The solution can be improved as shown in ***Tables 6·42*** and ***6·43***.

Table 6·42

5 • (2)			5
	√ + θ	8 − θ (1)	8
	7 • (4)		7
2 • (1)	2 − θ (6)	10 + θ (2)	14
7	9	18	

Table 6·43

5(2)			5
	2(3)	6(1)	8
	7(4)		7
2(1)		12(2)	14
7	9	18	

Here min $[8-\theta, 2-\theta] = 0$ or $2-\theta = 0$ or $\theta = 2$ units.

Table 6·44

Matrix for set of u_i and v_j

			u_i
(2)			1
	(3)	(1)	−1
	(4)		0
(1)		(2)	0
v_j: 1	4	2	

Table 6·45

Matrix $[c_{ij}]$ for empty cells

•	(7)	(4)
(3)	•	•
(5)	•	(7)
•	(6)	•

Table 6·46

Matrix $[u_i + v_j]$ for empty cells

•	5	3
0	•	•
1	•	2
•	4	•

Table 6·47

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	2	1
3	•	•
4	•	5
•	2	•

Since all empty cell evaluations in *Table 6·47* are positive, the solution in *Table 6·43* is optimal. The minimum cost for this solution is = 5(2) + 2(3) + 6(1) + 7(4) + 2(1) + 12(2) = Rs. 76.

Example 5. *Solve the transportation problem :*

	D_1	D_2	D_3	D_4	Available
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Required	20	40	30	10	100 Total

Solution. By ***'Lowest Cost Entry Method'***, the starting basic feasible solution is obtained having the transportation cost Rs 180.

Table 6·48

					Available
	20(1)		10(1)		30
		20(3)	20(2)	10(1)	50
		20(2)			20
Requirement	20	40	30	10	

To test this solution for optimality, following tables are obtained :

Table 6·49

Matrix for set of u_i and v_j

				u_i
(1)		(1)		−1
	(3)	(2)	(1)	0
	(2)			−1
v_j: 2	3	2	1	

Table 6·50

Matrix $[c_{ij}]$ for empty cells only

•	(2)	•	(4)
(3)	•	•	•
(4)	•	(5)	(9)

Table 6·51

Matrix $(u_i + v_j)$ for empty cells

•	2		0
2	•	•	•
1	•	1	0

Table 6·52

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	0	•	4
1	•	•	•
3	•	4	9

Table 6·52 shows that all empty cell evaluations are non-negative. Hence the solution under test is optimal.

Furthermore, the cell evaluation $d_{12} = 0$ indicates that alternative optimal solutions also exist.

Example 6. *Determine the optimum basic feasible solution to the following transportation problem.*

	To A	B	C	Available
I	50	30	220	1
	90	45	170	3
III	250	200	50	4
Required	4	2	2	

[I.A.S. (Main) 91]

Solution. The intial basic feasible solution can be easily obtained by two different methods as follows :

By Lowest Cost Method

	1 (30)		1
2 (90)	1 (45)		3
2 (250)		2 (50)	4
4	2	2	

Cost $z = 1 \times 30 + 2 \times 90 + 1 \times 45 + 2 \times 250 + 2 \times 50$
= Rs. 855.

By Vogel's Method

1 (50)	(30)	220	1
3 (90)	(45)	(170)	3
(250)	2 (200)	2 (50)	4
4	2	2	

Cost $z = 1\ (50) + 3\ (90) + 2\ (200) + 2\ (50)$
= Rs. 820.

Proceeding to test the initial solution (by *lowest cost method)* for optimality, following tables are obtained :

Matrix for u_i and v_j

				u_i
		(30)		−15
	(90)	(45)		0
	(250)		(50)	160
v_j	90	45	−110	

Matrix $[c_{ij}]$ for empty cells

(50)	•	(220)
•	•	(170)
•	(200)	•

Matrix $(u_i + v_j)$ for empty cells

				u_i
	75	•	−125	−15
	•	•	−110	0
	•	205	•	160
v_j	90	45	−110	

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

−25	•	345
•	•	280
•	−5	•

Since cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ are not all non-negative, the solution under test is not optimal.

First Iteration. Since the most negative cell evaluation is $d_{11} = -25$, so allocate as much as possible to cell (1, 1). This necessitates the shifting of 1 unit to this cell (1, 1) from cell (1, 2) as directed by the closed loop in the following table. Thus, cell (1, 1) enters the solution, while cell (1, 2) leaves the solution, *i.e.*, it becomes empty.

Table 6·53

+θ √	1 − θ (30)		1
2 − θ (90)	1 + θ (45)		3
2 • (250)		2 • (50)	4
4	2	2	

Improved solution (z = Rs. 830)

1(50)			1
1(90)	2(45)		3
2(250)		2(50)	4
4	2	2	

Here min $[1 - \theta, 2 - \theta] = 0$ or $1 - \theta = 0$ or $\theta = 1$ unit.

To test the improved solution for optimality :

Matrix for u_i and v_j

			u_i
(50)			50
(90)	(45)		90
(250)		(50)	250
v_j 0	−45	−200	

Matrix $[c_{ij}]$ for empty cells

•	(30)	(220)
•	•	(170)
•	(200)	•

Matrix $(u_i + v_j)$ for empty cells

			u_i
•	5	−150	50
•	•	−110	90
•	205	•	250
v_j 0	−45	−200	

Matrix for $[c_{ij} - (u_i + v_j)]$

•	25	370
•	•	280
•	−5(√)	•

Since the cell evaluation, $d_{32} = -5$ (negative), the improved solution under test is not optimal. So proceed to second iteration.

Second Iteration. Since the largest negative cell evaluation is $d_{32} = -5$, so allocate as much as possible to cell (3, 2). Thus, the cell (3, 1) leaves the solution while (3, 2) enters the solution. Cell (2, 2) may also be selected for leaving.

1 • (50)			1
$1+\theta$ (90)	$2-\theta$ (45)		3
$2-\theta$ (250)	$+\theta$ √		4
4	2	2	

Improved solution (z = Rs. 820)

1(50)			1
3(90)	0(45)		3
	2(200)	2(50)	4
4	2	2	

Here $2 - \theta = 0$ or $\theta = 2$ units.

Again, proceed as earlier to test the next improved solution for optimality. It has been observed that all cell evaluations are now non-negative. Hence the solution under test is optimal.

$$z = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50 = \text{Rs. } 820.$$

This solution was initially obtained by *Vogel's method.*

Example 7. *Determine the optimum basic feasible solution to the following transportation problem :*

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24 (Total)

where O_i and D_j denote ith *origin and* jth *destination, respectively.*

Solution. The initial solution to this problem can be easily obtained. In order to make the number of allocations equal to $m + n - 1$, allocate 0 amount to cell (3, 2), otherwise it is not possible to test for optimality Also see **sec 10.11** on ***degeneracy*** for handling such situations. $z = 12 + 4 + 0 + 12 + 0 + 0 = \text{Rs. } 28.$

	6(2)			6
		2(2)	6(0)	8
4 (0)	0(2)	6(2)		10
4	6	8	6	

To test the initial solution for optimality (short-cut way). Compute the following table. Once the process is understood carefully, it will be easy to compute all the necessary information in one table only, instead of computing four tables for $[(u_i, v_j), (c_{ij}), (u_i + v_j), c_{ij} - (u_i + v_j)]$.

				u_i
1 (1) 0+0	6 • (2)	1 (3) 0+2	4 (4) 0+0	0
4 (4) 0+0	1 (3) 0+2	2 • (2)	6 • (0)	0
4 • (0)	0 • (2)	6 • (2)	1 (1) 0+0	0
v_j: 0	2	2	0	

Each cell of above table is read as follows :

1. The numbers written at the centre of the (*occupied*) cells are the allocations x_{ij} in the current solution. Occupied cells are also called ***basic-cells*** indicated by '•' at the centre of the basic cells.
2. **Down-left corner.** The numbers in the parenthesis written at the down-left corners of the cells are the unit transportation costs (c_{ij}).
3. **Down-right corner.** The numbers written at the down-right corners of (*empty*) cells are the *sum* of numbers u_i and v_j [*i.e.*, $u_i + v_j$]. The numbers u_i and v_j are calculated so that $c_{ij} = u_i + v_j$ for each *occupied cell*.
4. **Upper-right corner.** The numbers written at the upper-right corners of each *empty* cell are the net evaluations $d_{ij} = (c_{ij}) - (u_i + v_j)$ which are simply obtained by subtracting the number $u_i + v_j$ (at the down-right corner) from the corresponding unit cost (c_{ij}) written at the down-left corner. The cell evaluations are computed as follows :

$d_{11} = c_{11} - (u_1 + v_1) = (1) - (0+0) = 1$ $\quad$ $d_{14} = c_{14} - (u_1 + v_4) = (4) - (0+0) = 4$

$d_{21} = c_{21} - (u_2 + v_1) = (4) - (0+0) = 4$ $\quad$ $d_{22} = c_{22} - (u_2 + v_2) = (3) - (0+2) = 1$

$d_{13} = c_{13} - (u_1 + v_3) = (3) - (0+2) = 1$ $\quad$ $d_{34} = c_{34} - (u_3 + v_4) = (1) - (0+0) = 1$

These calculations can be done within the table orally.

From these calculations, it is observed that all cell evaluations (d_{ij}) are positive.

Hence the solution under test is optimal.

Example 8. *Given the following data.*

The cost of shipment from third source to the third destination is not known. How many units should be transported from sources to the destinations so that the total cost of transporting all the units to their destinations is a minimum.

		Destinations			
		1	2	3	Capacity
Sources	*1*	2	2	3	10
	2	4	1	2	15
	3	1	3	×	40
Demand		20	15	30	

Solution. Since the cost c_{33} is unknown, assign a large cost, say M, to this cell. Now using *Vogel's method* an initial basic feasible solution is obtained.

Table 6·54
Initial B.F.S.

(2)	(2)	10(3)
(4)	(1)	15(2)
20(1)	15(3)	5(M)

Table 6·55
$(u_i + v_j)$ Matrix

			u_i
4 − M	6 − M	• (3)	3 − M
3 − M	5 − M	• (2)	2 − M
• (1)	• (3)	• (M)	0
v_j: 1	3	M	

Table 6·56

Matrix $[c_{ij} - (u_i + v_j)]$

				u_i
	$2-(4-M)$ + ive	$2-(6-M)$ + ive	•	$3-M$
	$4-(3-M)$ + ive	$1-(5-M)$ + ive	•	$2-M$
	•	•	•	0
v_j	1	3	M	

Since all the net evaluations for empty cells are positive, the current solution is optimum. But, it is to be noticed here that cell (3, 3) also appears in the solution for which the cost for shipment is not known. Hence, there exists a ***pseudo optimum basic feasible solution*** : $x_{13} = 10$, $x_{23} = 15$, $x_{31} = 20$, $x_{23} = 15$ and $x_{33} = 5$.

Example 9. *Is* $x_{13} = 50, x_{14} = 20, x_{21} = 55, x_{31} = 30, x_{32} = 35, x_{34} = 25$ *an optimum solution of the following transportation problem ?*

		To 1	2	3	4	Available units
	I	6	1	9	3	70
From	II	11	5	2	8	55
	II	10	12	4	7	90
Required units		85	35	50	45	

If not, modify it to obtain a between feasible solution. **[Meerut (Maths) 91]**

Solution. The initial feasible solution is given as in the following table with cost of Rs. 1460.

		50(9)	20(3)
55(11)			
30(10)	35(12)		25(7)

Matrix for u_i and v_j

					u_i
			• (9)	• (3)	−4
	• (11)				1
	• (10)	• (12)		• (7)	0
v_j	10	12	13	7	

Starting Table. *Vacate the cell (3, 4) and occupy (2, 3).* [θ = min (50 , 55 , 25) = 25]

					u_j
	0 (6) 6	−7 (1) 8	50 − θ (9)	20 + θ (3)	−4
	55 − θ (11)	−8 (5) 13	−12 +θ√ (2) 14	0 (8) 8	1
	30 + θ (10)	35 (12)	−9 (4) 13	25 − θ (7)	0
v_j	10	12	13	7	

First Iteration Table. *Vacate the cell (1, 3) and occupy (1, 2).* Here θ = min [35 , 30 , 25] = 25.

					u_i
	−12 (6) 18	+ θ −19 √ (1) 20	25 − θ (9)	45 • (3)	0
	30 − θ (11)	−8 (5) 13	25 + θ (2)	12 (8) −4	−7
	55 + θ (10)	35 − θ (12)	3 (4) 1	3 12 (7) −5	−8
v_j	18	20	9	3	

Second Iteration Table. *Vacate cell (2, 1) and occupy (2, 2).* Here $\theta = \min(5, 10) = 5$.

				u_i
7 (6) −1	25 • (1)	19 (9) −10	45 • (3)	0
5 − θ (11)	θ √ −8 (5) 13	50 • (2) 14	−7 (8) 15	12
80 + θ (10)	10 − θ (12)	3 (4) 1	−7 (7) 14	11
v_j −1	1	−10	3	

Third Iteration Table. *Vacate the cell (3, 2) and* occupy (3, 4). Here $\theta = \min[45, 5] = 5$.

				u_i
7 (6) −1	25 + θ (1)	11 (9) −2	45 − θ (3)	0
8 (11) 3	• 5 (5)	50 • (2)	1 (8) 7	4
85 • (10)	5 − θ (12)	−5 (4) 9	−7 √ + θ (7) 14	11
v_j −1	1	−2	3	

Since all the net evaluations are non-negative, an optimum basic feasible solution is obtained as :
$x_{12} = 30$, $x_{14} = 40$, $x_{22} = 5$, $x_{23} = 50$, $x_{31} = 85$, and $x_{34} = 5$.
$z = 30 \times 1 + 40 \times 3 + 5 \times 5 + 50 \times 2 + 85 \times 10 + 5 \times 7$
= Rs. 1160 .

Optimal Iteration Table

				u_i
0 (6) 6	30 • (1)	11 (9) −2	40 • (3)	0
1 (11) 10	5 • (5)	50 • (2)	1 (8) 7	4
85 • (10)	7 (12) 5	2 (4) 2	5 • (7)	4
v_j 6	1	−2	3	

Example 10. *Solve the transportation problem where all entries are unit costs.*

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	73	40	9	79	20	8
O_2	62	93	96	8	13	7
O_3	96	65	80	50	65	9
O_4	57	58	29	12	87	3
O_5	56	23	87	18	12	5
b_j	6	8	10	4	4	

Solution. Using ***'Lowest Cost Entry Method,*** the initial basic feasible solution having transportation cost Rs. 1123 is obtained as below.

		8(9)			8
3(62)			4(8)		7
2(96)	7(65)				9
1(57)		2(29)			3
	1(23)			4(12)	5
6	8	10	4	4	

Starting Iteration Table

					u_i
	(73) + 37	(40) + 6	(9) 8	(79) + −17	(20) + −5	−25
	(62) 3 − θ	(93) + 31	(96) + 34	(8) 4	(13) −7 + θ √ 20	0
	(96) 2 + θ	(65) 7 − θ	(80) + 68	(50) + 42	(65) + 54	34
	(57) 1	(58) + 26	(29) 2	(12) + 3	(87) + 15	−5
	(56) + 54	(23) 1 + θ	(87) + 26	(18) + 0	(12) 4 − θ	−8
v_j	62	31	34	8	20	

Here θ = min [3, 4, 7] = 3.

First Iteration Table. *Vacate the cell (2, 1) and occupy (2, 5).*

						u_i
	(73) + 37	(40) + 6	(9) 8	(79) + −10	(20) + −5	−18
	(62) + 55	(93) + 24	(96) + 27	(8) 4	(13) 3	0
	(96) 5	(65) 4	(80) + 68	(50) + 49	(65) + 54	41
	(57) 1	(58) + 26	(29) 2	(12) + 10	(87) + 15	2
	(56) + 54	(23) 4	(87) + 26	(18) + 7	(12) 1	−1
v_j	55	24	27	8	13	

Since all the net evaluations at the upper-right corners of all empty cells are non-negative, the solution under test is optimal. Hence optimum allocation is given by, $x_{13} = 8$, $x_{24} = 4$, $x_{25} = 3$, $x_{31} = 5$, $x_{32} = 4$; $x_{41} = 1$, $x_{43} = 2$, $x_{52} = 4$ and $x_{55} = 1$, and minimum transportation cost

$$z = 8 \times 9 + 4 \times 8 + 3 \times 13 + 5 \times 96 + 4 \times 65 + 1 \times 57 + 2 \times 29 + 4 \times 23 + 1 \times 12 = \text{Rs. } 1102.$$

Example 11. *The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.*

		Market				
		I	*II*	*III*	*IV*	Supply
Warehouse	A	5	2	4	3	22
	B	4	8	1	6	15
	C	4	6	7	5	8
Requirement		7	12	17	9	

The shipping clerk has worked-out the following schedule from experience : 12 units from A to II, 1 unit from A to III, 9 units from A to IV, 15 units from B to III, 7 units from C to I, and 1 unit from C to III.

(a) *Check and see if the clerk has the optimal schedule,*

(b) *Find the optimal schedule and minimum total shipping cost.*

(c) *If the clerk is approached by a carrier of route C to II who offers to reduce his rate in the hope of getting some business, by how much must the rate be reduced before the clerk should consider giving him an order.* **[VTU (BE Mech.) 2003]**

Solution. The basic feasible solution is given as follows :

	12 (2)	1 (4)	9 (3)	22
		15 (1)		15
7 (4)		1 (7)		8
7	12	17	9	

(a) Starting Iteration Table.

				u_i
+ (5) 1	12 • (2)	1 + θ • (4)	9 − θ • (3)	0
6 (4) −2	+ (8) −1	• 15 (1)	6 (6) 0	− 3
7 • (4)	+ (6) 5	1 − θ • (7)	−1 + θ √ (5) 6	3
v_j 1	2	4	3	

(b) Optimal Table

				u_i
+ (5) 2	12 • (2)	2 • (4)	8 • (3)	0
+ (4) − 1	+ (8) − 1	15 • (1)	6 (6) 0	− 3
7 • (4)	+ (6) 4	+ (7) 6	1 • (5)	2
v_j 2	2	4	3	

Here $\theta = \min [9, 1] = 1$.

Since all the net evaluations (at the upper-right corners) of empty cells are non-negative, the current solution is optimal.

Hence the optimal schedule is : $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$.

Minimum total shipping cost will be $z = 12 \times 2 + 2 \times 4 + 8 \times 3 + 15 \times 1 + 7 \times 4 + 1 \times 5 =$ Rs. 104.

(c) If the clerk decides to transport at the most 8 units from *C* to *II* (instead of 7 to *I* and 1 to *IV*), then *II* may reduce his cost from $c_{32} = 6$ to at least 4 in order to have the improved cost. So according to the given proposal, the total minimum transportation cost will become Rs. 103.

Example 12. *The following table gives the cost for transporting material from supply points A, B, C and D to demand points E, F, G, H, and J.*

		To				
		E	F	G	H	J
From	*A*	8	10	12	17	15
	B	15	13	18	11	9
	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows :

A to E 90 ; A to F 10 ; B to F 150 ; C to F 10 ; C to G 50 ; C to J 120 ; D to H 210 ; D to J 70.

(a) *Check if this allocation is optimum. If not, find an optimum schedule.*

(b) *If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum schedule ?* **[IAS (Main) 91]**

Solution. The given allocation has the transportation cost = Rs. 6930. Now test the given solution for optimality as follows :

Starting Iteration Table

					u_i
90 − θ (8)	10 + θ (10)	+ (12) −4	+ (17) −3	+ (15) 3	10
+ (15) 11	+ 150 (13)	+ (18) −1	+ 11 0	+ 9 6	13
−4 θ √ (14) 18	10 − θ (20)	50 (6)	+ (10) 7	120 (13)	20
−4 (13) 17	0 (19) 19	+ (7) 5	210 (6)	70 (12)	19
v_j: −2	0	−14	−13	−7	

Here θ = min [90 , 10] = 10.

First Iteration Table. *Vacate the cell (3, 2) and occupy (3, 1).*

					u_i
80 − θ (8)	20 + θ (10)	+ (12) 0	+ (17) 1	+ 15 7	−6
+ (15) 11	150 − θ (13)	+ (18) 3	+ 11 4	−1 + θ √ (9) 10	−3
10 + θ (14)	+ (20) 16	50 (6)	(10) 7	120 − θ (13)	0
0 (13) 13	+ (19) 15	+ (7) 5	210 (6) 5	70 (12)	−1
v_j: 14	16	6	7	13	

Here θ = min [120, 80, 150] = 80,

Second Iteration (Optimal) Table

					u_i
(8) 7 +	(10) 100 ●	(12) −1 +	(17) 0 +	(15) 6 +	−3
(15) 10 +	(13) 70 ●	(18) 2 +	(11) 3 +	(9) 80 ●	0
(14) 90 ●	(20) 17 +	(6) 50 ●	(10) 7 +	(13) 40 ●	4
(13) 13 0	(19) 16 +	(7) 5 +	(6) 210 ●	(12) 70 ●	3
v_j: 10	13	2	3	9	

Since all the net-evaluations are non-negative, the following optimum solution is obtained :

$$x_{12} = 100, x_{22} = 70, x_{25} = 80, x_{31} = 90, x_{33} = 50, x_{35} = 40, x_{44} = 210, x_{45} = 70,$$

and minimum cost = Rs. 6810.

Example 13. *Hindustan Construction Company needs 3, 3, 4 and 5 million cubic feet of fill at four earthen dam-sites in Punjab. It can transfer the fill from three mounds A, B and C where 2, 6 and 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the table.*

(a) Solve the problem using transportation algorithm for minimum cost :

(b) *Formulate the problem as L.P.P.*

		To				
		I	II	III	IV	a_i
From	A	15	10	17	18	2
	B	16	13	12	13	6
	C	12	17	20	11	7
	b_j	3	3	4	5	

Solution. Using "*Vogel's Method*" the initial B.F.S with cost Rs. 190 is obtained as follows :

		2(17)	
	3(13)	2(12)	1(13)
3(12)			4(11)

Starting Iteration Table

				u_i
(15) 19 −4	(10) 18 −8 +θ √	(17) 2 − θ	(18) 18 0	5
(16) 14 +	(13) 3 − θ	(12) 2 + θ	(13) 1 ●	0
(12) 3 ●	(17) 11 +	(20) 10 +	(11) 4 ●	−2
v_j: 14	13	12	13	

Hence θ = min [3, 2] = 2.

Since all the net-evaluations are not non-negative, the current solution is not optimal.

First Iteration Table. *Vacate the cell (1, 3) and occupy (1, 2).*

					u_i
	(15) 11 +	2 • (10)	(17) 9 +	(18) 10 +	−3
	(16) 14 +	1 • (13)	4 • (12)	1 • (13)	0
	3 • (12)	(17) 11 +	(20) 10 +	4 • (11)	−2
v_j	14	13	12	13	

Since all the net-evaluations are non-negative, the optimum solution is

$$x_{12} = 2, x_{22} = 1, x_{23} = 4, x_{24} = 1, x_{31} = 3, x_{34} = 4,$$

and minimum cost z = Rs. 174.

L.P. Formulation. Let x_{ij} be the amount of fill transferred from mounds to dam-sites. Then formulation of the problem becomes:

$$\text{Min } z = (15x_{11} + 10x_{12} + 17x_{13} + 18x_{14}) + (16x_{21} + 13x_{22} + 12x_{23} + 13x_{24}) + (12x_{31} + 17x_{32} + 20x_{33} + 11x_{34})$$

subject to the constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} = 2 \qquad x_{12} + x_{22} + x_{32} = 3$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 6 \qquad x_{13} + x_{23} + x_{33} = 4$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 7 \qquad x_{14} + x_{24} + x_{34} = 5$$
$$x_{11} + x_{21} + x_{31} = 3 \qquad \text{and all } x_{ij} \geq 0 \ (i = 1, 2, 3; j = 1, 2, 3, 4)$$

EXAMINATION PROBLEMS

1. Find the optimum solution to the following transportation problem for which the cost, origin-availabilities, and destination-requirements are as given below:

(i)

To

		A	B	C	D	E	a_i
	I	3	4	6	8	8	20
	II	2	10	1	5	30	30
From	*III*	7	11	20	40	15	15
	IV	2	1	9	14	18	13
	$b_j \rightarrow$	40	6	8	18	6	

[Meerut M.Sc (Maths.) Jan. 98 BP, 94]

[Hint. First find the starting solution by "Vogel's method" and then show that this is an optimal solution].

[Ans. $x_{11} = 14, x_{15} = 6, x_{21} = 4, x_{23} = 8, x_{24} = 18, x_{31} = 15, x_{41} = 7, x_{42} = 6$; min cost = Rs. 321.]

(ii)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	
O_1	1	2	1	4	5	2	30	
O_2	3	3	2	1	4	3	50	
O_3	4	2	5	9	6	2	75	
O_4	3	1	7	3	4	6	20	
b_j	20	40	30	10	50	25	175	Total

[Hint. Find the initial solution by 'Lowest Cost Entry Method'. Although, "Vogel's method" will also yield the same initial solution, but will be longer one. One improvement is only required to reach the optimal solution.]

[Ans. $x_{11} = 20, x_{13} = 10, x_{23} = 20, x_{24} = 10, x_{25} = 20, x_{32} = 40, x_{35} = 10, x_{36} = 25, x_{45} = 20$, min cost = Rs. 430.]

(iii)

Store

		A	B	C	a_i
Factory	I	10	9	8	8
	II	10	7	10	7
	III	11	9	7	9
	IV	12	14	10	4
	$b_j \rightarrow$	10	10	8	

[**Hint.** Find initial BFS by 'Lowest Cost Entry Method' and prove it to be optimal]

[**Ans.** $x_{11} = 6$,. $x_{12} = 2$, $x_{22} = 7$, $x_{32} = 1$, $x_{33} = 8$, $x_{41} = 4$, min. cost = Rs. 240]

(iv)

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	−2	3	70
O_2	2	4	0	1	38
O_3	1	2	−2	5	32
Demand	40	28	30	32	

[**Hint.** Use 'VAM' to find initial BFS with cost Rs. 86.]

[**Ans.** $x_{11} = 40$, $x_{12} = 26$, $x_{14} = 4$, $x_{24} = 38$, $x_{32} = 2$ and $x_{33} = 30$; min cost = Rs. 86].

(v)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
$b_j \rightarrow$	16	18	31	25	

[**Hint.** Find initial BFS by 'VAM' and prove it to be optimal.

[**Ans.** $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 30$, min. cost = Rs. 355]

(vi)

To

		1	2	3	4	5	Supply
From	1	2	3	5	7	5	17
	2	4	1	2	1	6	13
	3	2	8	6	1	3	16
	4	5	3	7	2	4	20

[**Hint.** By VAM we get degeneracy. Find initial solution by *Lowest Cost Entry Method*, then prove optimum.]

(vii)

To

		1	2	3	a_i
From	I	2	7	4	5
	II	3	3	7	8
	III	5	4	1	7
	IV	1	6	2	14
	$b_j \rightarrow$	7	9	18	

[Meerut (B. Sc.) 90]

[**Hint.** Find initial solution by VAM and and revise once optimum.]

[**Ans.** $x_{11} = 5$, $x_{22} = 8$, $x_{32} = 1$, $x_{33} = 6$, $x_{41} = 2$, $x_{43} = 12$, min. cost = Rs. 70]

(viii)

To

		1	2	3	4	Supply
From	1	5	3	6	4	30
	2	3	4	7	8	15
	3	9	6	5	8	15
Demand		10	25	18	7	60 (Total)

[**Hint.** Find initial solution by VAM and it will be proved optimum. to get. Alternative solutions also exist.]

[**Ans.** (i) $x_{12} = 20$, $x_{13} = 3$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 5$, $x_{33} = 15$, min. cost = Rs. 231 (ii) $x_{12} = 23$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 15$

(ix)

	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Demand	22	35	25	41	123 (Total)

[**Hint.** Find initial solution by VAM, it will be proved optimum]

[**Ans.** $x_{14} = 30$, $x_{21} = 5$, $x_{22} = 35$, $x_{31} = 17$, $x_{33} = 25$, $x_{34} = 11$, cost = Rs. 1921

(x)

	D_1	D_2	D_3	D_4	Supply
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

[Mudurai BSc (Comp. Sc.) 92]

[**Ans.** $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 33$, min cost = 355]

(xi)

	D_1	D_2	D_3	D_4	s_i
O_1	3	8	9	16	8
O_2	6	11	14	6	9
O_3	5	13	10	12	13
d_j	6	7	7	10	

[**Ans.** $x_{11} = 4$, $x_{12} = 4$, $x_{24} = 9$, $x_{31} = 2$, $x_{32} = 3$, $x_{33} = 7$, $x_{34} = 1$, min cost = 256]

(xii)

	I	II	III	IV	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Requirement	7	12	17	9	

[**Ans.** $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$, min. cost = 149]

2. Solve the following cost-minimizing transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	30
Required	30	50	20	40	30	10	180

[Delhi B.Sc. (Maths.) 91]

[**Ans.** $x_{12} = 50$, $x_{23} = 20$, $x_{31} = 30$, $x_{34} = 20$, $x_{36} = 10$, $x_{44} = 20$, $x_{45} = 10$, min cost = 330]

3. There are three sources or origins which store a given product. These sources supply these products to four dealers. The capacities of the sources and the demands are as given below:

$$S_1 = 150,\ S_2 = 40,\ S_3 = 80\ \ D_1 = 90,\ D_2 = 70,\ D_3 = 50,\ D_4 = 60.$$

The cost of transporting the product from various sources to various dealers is shown in the table below.

	D_1	D_2	D_3	D_4
S_1	27	23	31	69
S_2	10	45	40	32
S_3	30	54	35	57

Find out the optimum solution for transporting the products at a minimum cost.

[**Hint.** Find initial BFS by VAM and prove it to be optimal.]

[**Ans.** $x_{11} = 30$, $x_{12} = 70$, $x_{13} = 50$, $x_{24} = 40$, $x_{31} = 60$, $x_{34} = 20$, min cost = 8190]

4. A firm manufacturing a single product has plants I, II, and III. The three plants have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 2 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D, and 30 units to customer E. Find the minimum possible transportation cost of shipping the manufactured product to five customers. The net per unit cost of transporting from the three plants to five customers is given in the table.

Customer

		A	B	C	D	E
	I	4	1	3	4	4
Plant	*II*	2	3	2	2	3
	III	3	5	2	4	4

[**Hint.** Find initial BFS by VAM and prove it to be optimal.]

[**Ans.** $x_{12} = 45$, $x_{15} = 15$, $x_{21} = 17$, $x_{24} = 18$, $x_{31} = 5$, $x_{33} = 20$, $x_{35} = 15$, min. cost = Rs. 290].

5. The figures in the body of the table below are proportional to the cost of transportation of the tonne of food grain from the port given by the row heading to the destination given by column heading.

Ports	Delhi	Hyderabad	Mysore	Nagpur	stock (in thousand tonnes)
Bombay	9	5	8	5	225
Calcutta	9	10	13	7	75
Madras	14	5	3	7	100
Requirements (thousand tonnes)	125	80	95	100	400

Plan a transportation scheme satisfying the requirements of each destination and at the same time minimizing the total transportation cost.

[**Hint.** Find the initial BFS by VAM and prove it to be optimal. Alternative solutions exists.]

[**Ans.** $x_{11} = 50$, $x_{12} = 75$, $x_{14} = 100$, $x_{21} = 75$, $x_{32} = 5$, $x_{33} = 95$, min. cost = 2310]

6. An oil corporation has got three refineries P, Q, and R and it has to send petrol to four different depots A, B, C and D. The cost of shipping 1 gal. of petrol and the available petrol at the refineries are given in the table. The requirement of the depots and the available petrol at the refineries are also given. Find the minimum cost of shipping after obtaining an initial solution by VAM.

		Depot				
		A	B	C	D	Available
	P	10	12	15	8	130
Refinery	Q	11	11	9	10	150
	R	20	9	7	18	170
Required		90	100	140	120	

[Hint. Find initial BFS by VAM and prove it to be optimal.]

[Ans. $x_{11} = 90$, $x_{14} = 40$, $x_{23} = 70$, $x_{24} = 80$, $x_{32} = 100$, $x_{33} = 70$; min. cost = Rs. 3460]

7. King Mahendra of the Pallavedynasty Red sculptors stationed at Kanchipuram, Mallapuram and Tiruchirapalli. There were 15 sculptors at Kanchipuram, 30 at Mallapuram and 5 at Tiruchirapalli. Rock-cut shrines had to be excavated and 12 sculptors were required at Mandagapattu, 20 at Pallavapuram, 8 at seevamangalam and 10 at Mahendravadi, give a solution which minimizes the cost of transport of the sculptors assuming that the cost of transport per mile is the same for all the routes. The distance in miles between the different places is given in the following table.

From	To			
	Mandagapatta	Pallavapuram	Seevaman-galam	Mahandravadi
Kanchipuram	60	40	50	20
Mallapuram	70	30	65	55
Tiruchirapalli	100	200	120	210

[Ans. $x_{13} = 5$, $x_{14} = 10$, $x_{21} = 7$, $x_{22} = 20$, $x_{23} = 3$, $x_{31} = 5$; min distance = 2235 miles.]

8. A company has factories at F_1, F_2 and F_3 which supply warehouses at W_1, W_2 and W_3. Weekly factory capacities are 200, 160 and 90 units respectivley. Weekly warehouses requirements are 180, 120 and 150 units respectivley. Unit shipping costs (in rupees) are as follows :

		Warehouse			
		W_1	W_2	W_3	Supply
	F_1	16	20	12	200
Factory	F_2	14	8	18	160
	F_3	26	24	16	90
	Demand	180	120	150	350

Determine the optimum distribution for this company to minimize shipping costs. **[Kerala (M. Com.) 97)**

[Ans. $F_1 : x_{12} = 140$, $x_{12} = 0$, $x_{13} = 60$, $F_2 : x_{21} = 40$, $x_{22} = 120$, $x_{23} = 0$

$F_3 : x_{31} = 0$, $x_{32} = 0$, $x_{33} = 90$ Total transportation cost = Rs. 5,920]

9. A wholesale company has three warehouses from which supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are :

Warehouse no.	**Supply (units)**	**Customer no.**	**Demand (units)**
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Conveniently, total supply at the warehouses is equal to total demand from the customers. The following table gives the transportation costs per unit shipment from each warehouse to each customer.

Warehouse	**Customer**			
	1	2	3	4
1	3	6	8	5
2	6	1	2	5
3	7	8	3	9

Determine what supplies to despatch from each of the warehouses to each customer so as to minimize overall transportation cost. **[AIMA (Dl. in Management) Dec. 1996]**

[Ans. $x_{11} = 15$, $x_{14} = 5$, $x_{22} = 19$, $x_{24} = 9$, $x_{33} = 13$, $x_{34} = 4$. Total transportation cost = Rs. 209.]

10. A manufacturer has distribution centres at X, Y, and Z. These centres have availability 40, 20 and 40 units of his product. His retail outlets at A, B, C, D and E requires 25, 10, 20, 30 and 15 units respectively. The transport cost (in rupees) per unit between each centre outlet is given below :

Distribution centre	Retail outlets				
	A	B	C	D	E
X	55	30	40	50	50
Y	35	30	100	45	60
Z	40	60	95	35	30

Determine the optimal distribution to minimize the cost of transportation. **[Poone Univ. (M.B.A.) 1997]**

[**Ans.** $x_{11} = 5$, $x_{12} = 10$, $x_{13} = 20$, $x_{14} = 5$, $x_{21} = 20$, $x_{44} = 29$, $x_{45} = 15$, Total transportation cost = Rs. 3650]

11. *ABC* limited has three production shops supplying a product ot five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The cost of transportation are as given below :

Shop	Warehouse					Capacity
	I	II	III	IV	V	
A	6	4	4	7	5	100
B	5	6	7	4	8	125
C	3	4	6	3	4	175
Requirement	60	80	85	105	70	400

The cost of manufacture of the product at different production shops is :

Shop	Variable cost	Fixed cost
A	14	7,000
B	16	4,000
C	15	5,000

Find the optimum quantity to be supplied from each shop to different warehouses at minimum total cost.

[Nagpur (M.B.A.) Nov. 98]

[**Ans.** $x_{12} = 15$, $x_{13} = 85$, $x_{22} = 20$, $x_{24} = 105$, $x_{31} = 60$, $x_{32} = 45$, $x_{35} = 70$ Min. total transportation cost = 7605]

12. A company has three factories at Amethi, Baghpat and Gwalior; and four distribution centres at Allahabad, Bombay, Calcutta and Delhi. With identical cost of production at the three factories the only variable cost involved is transportation cost. The production at the three factories is 5,000 tonnes; 6,000 tonnes; 2,500 tonnes respectively. The demand at four distribution centres is 6,000 tonnes; 4,000 tonnes; 2,000 tonnes and 1,500 tonnes respectively. The transportation costs per tonne from different factories to different centres are given below :

Factory	Distribution Centre			
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gvalior	2	5	4	5

Suggest the optimum transportation schedule and find the minimum cost of transportation.

[Gujarat (MBA) 96; Delhi (M.Com.) 95]

[**Ans.** $x_{11} = 3500$, $x_{12} = 1500$, $x_{22} = 2500$, $x_{23} = 2000$, $x_{24} = 1500$ and $x_{31} = 2500$, Min. transportation cost = Rs. 39,500.]

13. A company manufacturing air-coolers has two plants located at Bombay and Calcutta with a weekly capacity of 200 units and 100 units, respectively. The company supplies air coolers to its 4 show rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75,100, 100 and 30 units, respectively. The cost of transportation per unit (in Rs.) is shown in the following table :

	Ranchi	Delhi	Lucknow	Kanpur
Bombay	90	90	100	100
Calcutta	50	70	130	85

Plant the production programme so as to minimize the total cost of transportation. **[Rajasthan Univ. (M.Com.) 97]**

[**Ans.** $x_{12} = 75$, $x_{13} = 95$, $x_{14} = 30$, $x_{21} = 75$, $x_{22} = 25$, $x_{31} = 30$, $x_{32} = 10$, $x_{33} = 0$.

Min. transportation cost = Rs. 24750. Alternative optimum solution also exists.]

14. The *ABC* Tool Company has a sales force of 25 men who work out from three regional offices. The company produces four basic product lines of hand tools. Mr. Jain, the sales manager feels that 6 salesmen are needed to distribute product line 1, 10 salesmen to distribute product line 2, 4 salesmen for product line 3 and 5 salesmen for product line 4. The cost (in Rs.) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows :

	Product lines 1	2	3	4
Regional office A	Rs. 20	Rs. 21	Rs. 16	Rs. 18
Regional office B	17	28	14	16
Regional office C	29	23	19	20

At the present time, 10 salesmen are allocated to office A, 9 salesmen to office B, and 7 salesmen to office C. How many salesmen should be assigned from each office to sell each product line in order to minimize costs ? Identify alternate optimum solutions, if any. **[Delhi Univ. (M.B.A.,) April 98)**

[**Ans.** $x_{12} = 4$, $x_{13} = 1$, $x_{14} = 5$, $x_{21} = 6$, $x_{23} = 3$, $x_{32} = 6$, $x_{35} = 1$, Min. transportation cost = Rs. 472.]

15. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depto 3; and 440,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	**Company 2**	**Company 3**
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum transportation schedule. **[Delhi Univ. (M..B.A.,) Nov. 97)**

[**Ans.** $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, Min. transportation cost = Rs. 51,70,000.]

16. A company has three plants at A, B and C which supply to warehouses located at D, E, F, G and H. Weekly plant capacities are 200, 125 and 225 tons respectively. Weekly warehouses requirements are 75, 105, 130 155 and 85 tons respectively. Unit transportation cost matrix is given below :

		To D	R	F	G	H
	A	50	82	65	60	35
From	B	45	70	70	65	50
	C	80	45	75	60	60

Determine the optimum cost distribution pattern and also the minimum total cost. **[IES (Mech.) 2000]**

[**Hint** : Transportation cost matrix is

From \ To	D	E	F	G	H	Supply (tons)	
A	50	82	65	60	35	200	15
B	45	70	70	65	50	125	5
C	80	45	75	60	40	225	5
Demand	75	105	130	155	85	550 / 550	
Penalties	5	25	5	0	5		

Here Demand = Supply

Applying Voget's approximation method.

From \ To	D	E	F	G	H	Supply (Tons)	Penaltties	
A	(50)	(82)	180 • (65)	120 • (60)	(35)	200/0	15	15
B	75 • (45) IV	(70)	50 • (70)	(65)	(50)	125/50/0	5	5
C	(80)	105 • (45) I	(75)	35 • (60) III	85 • (40) II	225/120/35/0	5	20
Demand (Tons)	75/0	105/0	130/0	155/120	85/0			

6.11 DEGENERACY IN TRANSPORTATION PROBLEMS

The solution procedure for non-degenerate basic feasible solution with exactly $m+n-1$ strictly positive allocations in ***independent positions*** has been discussed so far. However, sometimes it is not possible to get such initial feasible solution to start with. Thus degeneracy occurs in the transportation problem whenever a number of occupied cells is less than $m+n-1$.

We recall that a basic feasible solution to an m-origin and n-destination transportation problem can have at most $m+n-1$ number of positive (non- zero) basic variables. If this number is exactly $m+n-1$, the BFS is said to *be non-degenerate;* and if less than $m+n-1$ the basic solution degenerates. It follows that whenever the number of basic cells is less than $m+n-1$, the transportation problem is a degenerate one.

Degeneracy in transportation problems can occur in two ways :

1. Basic feasible solutions may be degenerate from the initial stage onward.
2. They may become degenerate at any intermediate stage.

Q. 1. What is degeneracy problem in transportation ? What is its cause ? How it can be over come.

2. Explain briefly about unbalanced transportation problem and degenerate case in transportation problem. **[VTU (BE Mech.) 2002]**

3. When do you say a solution to a transportation problem is degenerate ? **[JNTU (BE Comp. Sc.) 2004]**

6·11-1 Resolution of Degeneracy During the Initial Stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to *one* or *more* of the empty cells so that a number of occupied cells becomes $m+n-1$. The cell containing this extremely small allocation is, of course, considered to be an occupied cell.

Rule : The extremely small quantity usually denoted by the Greek letter Δ (delta) [also sometimes by ε (epsilon)] is introduced in the *least cost* independent cell subject to the following assumptions. If necessary, two or more Δ's can be introduced in the least and second least cost independent cells.

1. $\Delta < x_{ij}$ for $x_{ij} > 0$.
2. $x_{ij} + \Delta = x_{ij} - \Delta$, $x_{ij} > 0$.
3. $\Delta + 0 = \Delta$.

*4. If there are more than one Δ's in the solution, $\Delta < \Delta'$, whenever Δ is *above* Δ'. If Δ and Δ' are in the same row, $\Delta < \Delta'$ when *A is to the left of* Δ'.

Q. What do you mean by non-degenerate basic feasible solution of a transportation problem.

For example, $50 + \Delta = 50$, and $200 - \Delta = 200$. Of course, if Δ is subtracted from itself the result is assumed to be zero. Following example will make the procedure clear.

Example 14. *A company has three plants A, B and C and three warehouses X, Y and Z. Number of units available at the plants is 60, 70 and 80, respectively. Demands at X, Y and Z are 50, 80 and 80, respectively. Unit costs of transportation are as follows :*

Table 6·56

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

What would be your transportation plan ? Give minimum distribution cost.

[[JNTU (BE Comp. Sc.) 2004; Garhwal 97]

Solution. *Step 1.* First, write the given cost-requirement *Table 10·56* in the following manner :

Q. State the transportation problem in general terms and explain the problem of degeneracy. **[IAS (Maths.) 97]**

Table 10·57

	X	Y	Z	Available
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Requirement	50	80	80	210 (Total)

Step 2. Using either the *"Lowest Cost Entry Method"* or *"Vogel's Approximation Method,* obtain the initial solution.

Step 3. Since the number of occupied cells in ***Table 6·58*** is 4, which does not equal to $m + n - 1$ (that is 5), the problem is degenerate at the very beginning, and so the attempt to assign u_i and v_j values to *Table 6·58* will not succeed. However, it is possible to resolve this degeneracy by addition of Δ to some suitable cell.

Table 6·58

				Available
			60(3)	60
	50(3)		20(9)	70
		80(3)		80
Requirement	50	80	80	

Generally, it is not possible to add Δ to any empty cell. But, Δ must be added to one of those empty cells which make possible the determination of a unique set of u_i and v_j. So choose such empty cell with careful judgement, for if the empty cell (1,1) is made an occupied cell (by addition of Δ) it will not be possible to assign u_i and v_j values. The allocations in cell (1, 1), (1, 3), (2, 1), and (2, 3) will become in non-independent (rather than independent) positions.

On the other hand, the addition of Δ to any of the cells (1, 2), (2, 2), (3, 1) and (3, 3) will enable us to resolve the degeneracy and allow to determine a unique set of u_i and v_j values. So proceed to resolve the degeneracy by allocating Δ to least cost independent empty cell (3, 3) as shown in ***Table 6·59.***

Table 6·59

				Available
			60(3)	60
	50(3)		20(9)	70
		80(3)	Δ (5)	80 + Δ = 80
Requirement	50	80	80 + Δ = 80	

Now proceed to test this solution for optimality.

Step 4. Values of u_i and v_j will be obtained as shown in ***Table 6·60.***

Once a unique set of u_i and v_j values has been determined, various steps of the transportation algorithm can be applied in a routine manner to obtain an optimal solution.

Table 6·60

				u_i
			• (3)	−2
	• (3)		• (9)	4
		• (3)	• (5)	0
v_j	−1	3	5	

Table 6·61

Matrix $[c_{ij}]$ for empty cells only

(8)	(7)	•
•	(8)	•
(11)	•	•

Table 6·62

Matrix $[u_i + v_j]$ for empty cells

−3	1	•
•	7	•
−1	•	•

Since all cell-evaluations in *Table* 6·63 are positive, the solution under test is optimal. The real total cost of subsequent solutions did not happen to change in the example after Δ was introduced. In general, this will not be the case. In as much as the infinitesimal quantity Δ plays only an auxiliary role and has no significance, it is removed when the optimal solution is obtained. Hence, the final answer is given in *Table 6·64*

Table 6·63

Matrix $[c_{ij} - (u_i + v_j)]$

11	6	•
•	1	•
12	•	•

Table 6·64

		60(3)
50(3)		20(9)
	80(3)	

Thus minimum cost is : 180 + 150 + 180 + 240 = Rs. 750.

6·11-2 Resolution of Degeneracy During the Solution Stages

The transportation problem may also become degenerate during the solution stages. This happens when most favourable quantity is allocated to the empty cell having the largest negative cell-evaluation resulting in simultaneous vacation of two or more of currently occupied cells. To resolve degeneracy, allocate Δ to one or more of recently vacated cells so that the number of occupied cells is $m+n-1$ in the new solution. This type of degeneracy can be explained below in ***Example 15.***

Example 15. *The cost-requirement table for the transportation problem is given as below :*

Table 6·65

	W_1	W_2	W_3	W_4	W_5	Available
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
Required	30	30	15	20	5	

Solution. By ***'North-West-Corner Rule'***, the non-degenerate initial solution is obtained in ***Table 6·66 :***

Table 6·66

	W_1	W_2	W_3	W_4	W_5	Available
F_1	30(4)	10(3)				40
F_2		20(2)	10(3)			30
F_3			5(6)	15(3)		20
F_4				5(5)	5(3)	10
Required	30	30	15	20	5	

Now, test this solution for optimality to get the following table in usual manner.

Table 6·67

Matrix for set of u_i and v_j

						u_i
	4	3				0
		2	3			−1
			6	3		2
				5	3	4
v_j	4	3	4	1	−1	

Table 6·68

Matrix $[c_{ij}]$ for empty cells

•	•	(1)	(2)	(6)
(5)	•	•	(4)	(5)
(3)	(5)	•	•	(2)
(2)	(4)	(4)	•	•

Table 6·69

Matrix (u_i+v_j) for empty cells

•	•	4	1	−1
3	•	•	0	−2
6	5	•	•	1
8	7	8	•	•

Table 6·70

Matrix $[c_{ij}-(u_i+v_j)]$ for empty cells

•	•	−3	1	7
2	•	•	4	7
−3	0	•	•	1
−6√	−3	−4	•	•

Since, the largest negative cell evaluation is $d_{41}=-6$ (marked √), allocate as much as possible to this cell (4, 1). This necessitates shifting of 5 units to this cell (4, 1) as directed by the closed loop in ***Table 6·71.***

Table 6·71

Here maximum possible value of θ is obtained by usual rule :

						Available
	(4) 30 − θ	(3) 10 + θ				40
		(2) 20 − θ	(3) 10 + θ			30
			(6) 5 − θ	(3) 15 + θ		20
	+ θ √			(5) 5 − θ	5 •	10
Required	30	30	15	20	5	

min. $[30-\theta, 20-\theta, 5-\theta, 5-\theta] = 0$ *i.e.*, $5-\theta = 0$ or $\theta = 5$ units.

From ***Table* 6·71**, the revised solution becomes :

Table 6·72

						Available
	25(4)	15(3)				40
		15(2)	15(3)			30
			0*(6)	20(3)		20
	5(2)			0*(5)	5(3)	10
Required	30	30	15	20	5	

In this solution, the number of allocations becomes less than $m+n-1$ on account of simultaneous vacation of two cells [(3, 3), (4, 4), as indicated by *]. Hence this is a degenerate solution.

Now, this degeneracy may resolve by adding Δ to one of the recently vacated cells [(3, 3) or (4, 4)]. But in minimization problem, add Δ to recently vacated cell (4, 4) only, because it has the lowest shipping cost of Rs 5 per unit.

The rest of the procedure will be exactly the same as explained earlier. This way, the optimal solution can be obtained.

Table 6·73

						Available
	5(4)		15(1)	20(2)		40
		30(2)				30
	15(3)				5(2)	20
	10(2)					10
Required	30	30	15	20	5	

Note. Here the aim of finding an initial solution by '*North-West Corner Rule*' is only to show as to how degeneracy may arise during the solution stages. Otherwise, the optimal solution can be obtained immediately by using the initial solution by Vogel's method.

Example 16. *Solve the following transportation problem* :

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
From O_3	6	5	9	12	3	11	2
O_4	6	8	11	2	2	10	9
	4	4	6	2	4	2	22 (Total)

[Meerut (Maths.) 97P, (Stat.) 90]

Solution. Using 'VAM' the initial basic feasible solution having the transportation cost Rs. 112 is given below :

Initial BFS

		5(9)				5
	4(3)	Δ (7)			2(5)	$6+\Delta=6$
1(6)		1(9)				2
3(6)			2(2)	4(2)		9
4	4	$6+\Delta=6$	2	4	2	

Since the number of allocations (= 8) is less than $m+n-1$ (= 9), a very small quantity Δ may be introduced in the independent cell (2, 3), although least cost independent cell is (2, 5).

Optimum Table

						u_i
(9) 6 +	(12) 5 +	(9) 5 •	(6) 2 +	(9) 2 +	(10) 7 +	2
(7) 4 +	(3) 4 •	(7) Δ •	(7) 0 +	(5) 0 +	(5) 2 •	0
(6) 1 •	(5) 5 0	(9) 1 •	(12) 2 +	(3) 2 +	(11) 7 +	2
(6) 3 •	(8) 5 +	(11) 9 +	(2) 2 •	(2) 4 •	(10) 7 +	2
v_j: 4	5	7	0	0	5	

Since all the net evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is : $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$, $x_{33} = 1$.

The optimum transportation cost is given by

$$z = 5(9) + 4(3) + \Delta\,(7) + 2(5) + 1(6) + 1(9) + 3(6) + 2(2) + 4(2) = \text{Rs. } 112 \quad (\text{since } \Delta \to 0).$$

Note. In above optimum table, Δ may also be introduced in least cost independent cell (2, 5).

Example 17. *Solve the following transportation problem (cell entries represent unit costs) :*

							Available
	5	3	7	3	8	5	3
	5	6	12	5	7	11	4
	2	1	3	4	8	2	2
	9	6	10	5	10	9	8
Required	3	3	6	2	1	2	17 (Total)

[Meerut (M.Com.) Jan. 98 (BP), (M.Sc.) 93 P]

Soluton. Using 'VAM', an initial BFS having the transportation cost Rs. 103 is given below :

Initial BFS

	1(3)				2(5)	3
3(5)			Δ (5)	1(7)		4 + Δ = 4
		2(3)				2
	2(6)	4(10)	2(5)			8
3	3	6	2 + Δ = 2	1	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1$ (= 9), introduce negligible quantity Δ in the independent cell (2, 4).

						u_i
(5) 2 +	(3) 1 •	(7) 7 0	(3) 2 +	(8) 4 +	(5) 2 •	−3
(5) 3 •	(6) 6 0	(12) 10 +	(5) Δ •	(7) 1 •	(11) 8 +	0
(2) −2 +	(1) −1 +	(3) 2 •	(4) −2 +	(8) 0 +	(2) 1 +	−7
(9) 5 +	(6) 2 •	(10) 4 •	(5) 2 •	(10) 7 +	(9) 8 +	0
v_j: 5	6	10	5	7	8	

Since all the net-evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is given by : $x_{12} = 1$, $x_{16} = 2$, $x_{21} = 3$, $x_{25} = 1$, $x_{33} = 2$, $x_{42} = 2$, $x_{43} = 4$ and $x_{44} = 2$.

The optimum transportation cost is

$$z = 1(13) + 2(5) + 3(5) + \Delta\,(5) + 1(7) + 2(3) + 2(6) + 4(10) + 2(5) = \text{Rs. } 103, \text{ as } \Delta \to 0.$$

Example 18. *A company has 4 warehouses and 6 stores; the cost of shipping one unit from warehouse i to store j is* c_{ij}.

If $C = (c_{ij}) = \begin{pmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{pmatrix}$, *and the requirements of the six stores are 4, 4, 6, 2, 4, 2 and quantities at the warehouse are 5, 6, 2, 9. Find the minimum cost solution.* **[IAS (Maths.) 95]**

Solution. Using *'lowest cost entry method'*, an initial solution having the transportation cost Rs 70 is obtained as below :

		5(7)				5
	4(1)	Δ (5)			2(3)	6 + Δ = 6
2(4)						2
2(4)		1(9)	2(0)	4(0)		9
4	4	6 + Δ = 6	2	4	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1$ (= 9), introduce a negligible quantity Δ in the independent empty cell (2, 3).

Starting Iteration Table

						u_i
(7) 2 +	(10) 3 +	(7) 5 •	(4) −2 +	(7) −2 +	(8) 5 +	−2
(5) 0 +	(1) 4 − θ	(5) Δ + θ	(5) −4 +	(3) −4	(3) 2 •	−4
(4) 2 − θ	(3) 5 √ + θ −2	(7) 9 −2	(9) 0 +	(1) 0 +	(9) 7 +	0
(4) 2 + θ	(6) 5 +	(9) 1 − θ	(0) 5 2 • +	(0) 4 •	(8) 7 +	0
v_j: 4	5	9	0	0	7	

Min. $[4 - \theta, 2 - \theta, 1 - \theta] = 0 \Rightarrow = \theta = 1$.

Since all the net-evaluations for the non-basic (empty) cells are not non-negative, the initial BFS is not optimal. The empty cell (3, 2) must be allocated the maximum possible amount θ = 1 to this cell. Consequently, cell (4, 3) becomes empty.

First Iteration Table. *Vacate the cell (4, 3) and occupy the cell (3, 2).*

						u_i
(7) 4 +	(10) 3 +	(7) 5 •	(4) 0 +	(7) 0 +	(8) 5 +	2
(5) 2 +	(1) 3 •	(5) 1 •	(5) −2 +	(3) −2 +	(3) 2 •	0
(4) 1 •	(3) 1 •	(7) 7 0	(9) 0 +	(1) 0 +	(9) 5 +	2
(4) 3 •	(6) 3 +	(9) 7 +	(0) 2 •	(0) 4 •	(8) 5 +	2
v_j: 2	1	5	−2	−2	3	

Since all the net evaluations are non-negative, the current solution is optimum. Hence the optimum solution is given by $x_{13} = 5, x_{22} = 3, x_{23} = 1, x_{26} = 2, x_{31} = 1, x_{32} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4$.

The optimum transportation cost is given by

$$z = 5(7) + 3(1) + 1(5) + 2(3) + 1(4) + 3(4) + 1(3) + 2(0) + 4(0) = \text{Rs. } 68.$$

Q. 1. Explain "Degeneracy" in a transportation problem. **[Bharathidasan B.Sc (Math) 90; Delhi B.Sc (Math.) 90]**

2. How does the problem of degeneracy arise in a transportation Problem ? Explain how does one overcome it. **[Meerut 94]**

3. Explain how to solve the degeneracy in transportation problems.

EXAMINATION PROBLEMS
(ON DEGENERACY)

1. ***Explain :*** (i) a method of constructing a basic solution of a transportation problem, (ii) the technique of improvement of the constructed basic solution. Solve the following transportation problem :

Cost-matrix

	To			Available
From	0	2	0	70
	1	4	0	30
	0	2	4	50
Required	70	50	30	150

[**Hint.** Use 'VAM' to find initial BFS and prove it to be optimal.]

[**Ans.** $x_{11} = 20, x_{12} = 50, x_{23} = 30, x_{31} = 50$; min. cost = Rs. 100].

2. Determine the optimal solution to each of the following degenerate transportation problem :

(i)

	D_1	D_2	D_3	D_4	D_5	$a_i \downarrow$
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
$b_j \rightarrow$	8	3	7	2	2	

[**Ans.** $x_{11} = 1, x_{13} = 1, x_{21} = 7, x_{15} = 2, x_{32} = 3, x_{33} = 6, x_{44} = 2$; min. cost = 56].

(ii) Demand points

		D_1	D_2	D_3	D_4	Supply
	S_1	2	3	11	7	6
Sources	S_2	1	0	6	1	1
	S_3	5	8	15	10	10
Demand		7	5	3	2	17

[**Ans.** $x_{12} = 5, x_{13} = 1, x_{24} = 1, x_{31} = 7, x_{33} = 2, x_{34} = 1$.
min cost = Rs. 102. Alternative solutions also exist.]

(iii)

	To				$a_i \downarrow$
From	10	7	3	6	3
	1	6	7	3	5
	7	4	5	3	7
$b_j \rightarrow$	3	2	6	4	

[**Ans.** $x_{13} = 3, x_{21} = 3, x_{24} = 2, x_{32} = 2, x_{33} = 3, x_{34} = 2$
min,. cost Rs. 47].

3. A manufacturer has distribution centres located at Agra, Allahabad and Calcutta. These centres have available 40, 20 and 40 units of his product. His retail outlets require the following number of units : *A*, 25; *B*, 10; *C*, 20; *D*, 30; *E*, 15. The shipping cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets				
	A	B	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Calcutta	40	60	95	35	30

Determine the optimal shipping cost.

[**Ans.** $x_{12} = 10, x_{13} = 20, x_{15} = 10, x_{21} = 20, x_{31} = 5, x_{34} = 30, x_{35} = 5$, min. cost = Rs. 3600].

4. A manufacturer wants to ship 8 loads of his product as shown in the table. The matrix gives the mileage from origin O to destination D. Shipping costs are Rs. 10 per load per mile. What shipping schedule should be used.

[**Hint.** Find the initial solution by using Vogel's method. In this solution, number of allocations is less than $m + n - 1$ (*i.e.*, 5). Hence resolve the degeneracy by introducing Δ to one of the empty cells [say, (2, 2)]. Then this initial solution will be optimal with minimum mileage 820 or cost Rs. 8200].

[**Ans.** $x_{11} = 1, x_{21} = 3, x_{32} = 2, x_{33} = 2$]

	D_1	D_2	D_3	Available
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
Required	4	2	2	

[IAS (Main) 91]

5. (i) Using *North-West Corner Rule* for initial basic feasible solution, obtain an optimum basic feasible solution to the following degenerate transportation problems :

	To			Available
From	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

[**Ans.** $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$, min. cost = 33]

(iii)

To	D_1	D_2	D_3	a_i
From O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5

[**Ans.** $x_{21} = 5, x_{22} = 5, x_{23} = 5, x_{33} = 5$ and min. cost = 35]

6. Solve the transportation problem whose cost matrix is given below.

	To				Available
From	5	5	4	7	5
	6	4	1	2	5
	5	9	1	4	6
	8	3	2	4	4
	6	5	3	1	6
Required	5	8	3	10	

[**Ans.** $x_{11} = 2, x_{12} = 3, x_{22} = 1, x_{24} = 4, x_{31} = 3, x_{33} = 3, x_{42} = 4, x_{54} = 6$, and min. cost = 73]

7. Obtain the optimal solution by using the best starting solution.

	To				a_i
From	10	20	5	7	10
	13	9	12	8	20
	4	15	7	9	30
	14	7	1	0	40
	3	12	6	19	50
b_j	60	60	20	10	

[**Ans.** $x_{13} = 10, x_{22} = 20, x_{31} = 30, x_{42} = 20, x_{43} = 10, x_{44} = 10, x_{51} = 30, x_{52} = 20$ and min cost = 830]

8. A company has 4 warehouses and 6 stores., the surplus in the warehouses, the requirements of the stores and costs (in Rs) of transporting one unit of the commodity from warehouse i to the store j are given below. How should the commodity be transported so that the total transportation cost is a minimum ? Obtain the initial program by applying the north-west corner rule :

Warehouse	Store						Surplus
	1	2	3	4	5	6	
1	7	5	9	5	10	7	30
2	7	8	24	7	9	13	40
3	4	10	5	6	10	4	20
4	11	8	12	7	12	11	80
Requrement	30	30	60	20	10	20	170

[Delhi B.Sc. (Math.) 90]

[**Ans.** $x_{12} = 10, x_{16} = 20, x_{21} = 30, x_{25} = 10, x_{33} = 20, x_{42} = 20, x_{43} = 40, x_{44} = 20$, and min. cost = Rs. 1370]

9. Given below is the unit costs array with supplies a_i, $i = 1, 2, 3$ and demands b_j, $j = 1, 2, 3, 4$,

		Sink				
		1	2	3	4	a_i
source	1	8	10	7	6	50
	2	12	9	4	0	40
	3	9	11	10	8	30
	b_j	25	32	40	23	120

Find the obtimal solution to the above Hitchcock problem. **[Meerut 2002]**

6.12 UNBALANCED TRANSPORTATION PROBLEMS

So far we have discussed the *balanced* type of transportation problems where the total destination requirement equals the total origin capacity (*i.e.*, $\Sigma a_i = \Sigma b_j$). But, sometimes in practical situations, the demand may be more than the availability or *vice versa* (*i.e.* $\Sigma a_i \neq \Sigma b_j$).

Thus, if in a transportation problem, the sum of all available quantities is not equal to the sum of requirements, that is, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then such problem is called an ***unbalanced transportation problem.***

6·12-1 To Modify Unbalanced T.P. to Balanced Type

An unbalanced T.P. may occur in two different forms. (i) *Excess of availability,* (ii) *Shortage in availability.*

We now discuss these two cases by considering our usual m-origin, n-destination T.P. with the condition that $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

Case 1. (Excess Availability, i.e. $\Sigma a_i \geq \Sigma b_j$).

The general T.P. may be stated as follows :

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij}, \text{ subject to the constraints}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n$$

$$\text{and} \qquad x_{ij} \geq 0, \qquad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

The problem will possess a feasible solution if $\Sigma a_i \geq \Sigma b_j$. In the first constraint, the introduction of slack variable $x_{i,\,n+1}$ $(i = 1, 2, \ldots, m)$ gives

$$\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} = a_i\,;\ i = 1, 2, \ldots, m$$

or $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} \right) = \sum_{i=1}^{m} a_i \quad \text{or} \quad \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) + \sum_{i=1}^{m} x_{i,\,n+1} = \sum_{i=1}^{m} a_i$$

or $$\sum_{i=1}^{m} x_{i,\,n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j = \textit{Excess of Availability.} \left(\because \sum_{i=1}^{m} x_{ij} = b_j \right)$$

If this *excess availability* is denoted by b_{n+1}, the modified general T.P. can be reformulated as :

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n+1} x_{ij}\,(c_{ij}), \text{ subject to the constraints :}$$

$$\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} = a_i, \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, \ldots, n+1$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j,$$

where $c_{i,\,n+1} = 0$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n+1} b_j$.

This is clearly the *balanced T.P.* and thus can be easily solved by transportation algorithm.

Working Rule : *Whenever $\Sigma a_i \geq \Sigma b_j$, we introduce a dummy destination-column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the difference $\Sigma a_i - \Sigma b_j$.*

Case 2. (Shortage in Availability, i.e. $\Sigma a_i \leq \Sigma b_j$):

In this case, the general T.P. becomes :

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij}) \text{ subject to the constraints :}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} \leq b_j, \qquad j = 1, \dots, n$$

$$x_{ij} \geq 0, \qquad i = 1, \dots, m; j = 1, \dots n.$$

Now, introducing the slack variable $x_{m+1,j}$ $(j = 1, \dots, n)$ in the second constraint, we get

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j, \qquad j = 1, \dots, n$$

or $$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij} + x_{m+1,j}\right) = \sum_{j=1}^{n} b_j \quad \text{or} \quad \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j$$

or $$\sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{n} a_i = \textit{shortage in availability } a_{m+1}, \text{ say.}$$

Thus the modified general T.P. in this case becomes :

$$\text{Minimize} \quad z = \sum_{i=1}^{m+1} \sum_{j=1}^{n} x_{ij} c_{ij},$$

subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i = 1, \dots, m+1$$

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j \quad , \quad j = 1, \dots, n$$

$$x_{ij} \geq 0 \quad , \quad i = 1, \dots, m+1; j = 1, \dots, n.$$

where $c_{m+1,j} = 0$ for $j = 1, \dots, n$ and $\sum_{i=1}^{m+1} a_i = \sum_{j=1}^{n} b_j$.

Working Rule : *Whenever $\Sigma a_i \leq \Sigma b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference $(\Sigma b_j - \Sigma a_i)$.*

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing a fictitious *sink* in the first case and a fictitious *source* in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to a sink represents the unfilled demand at that sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks (as the case may be) are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier. The method for dealing with such type of problems will be clear in *Example 19.*

Q. 1. What is unbalanced transportation problem ? How do you start in this case ?

2. Explain the technique used to solve the transportation problem with the following restrictions imposed separately in each problem :

(i) it is required to have all basic solutions non-degenerate,
(ii) it is required to have no allocation in the (i, j) th cell.
(iii) it is required to have some positive allocation in the (i, j) th cell.
(iv) it is found that the total units ready to ship be less than the total units required.

Example 19. *XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities :*

Warehouse location	Capacity (tonnes)
City A	90
City B	50
City C	80
City D	60

The warehouses supply tobacco to cigarette companies in three cities thay have the following demand :

Cigarette company	Demand (tonnes)
Bharat	120
Janta	100
Red Lamp	110

The adjoining railroad shipping costs per tonne (in hundred rupees) have been determined :

From \ To	Bharat	Janta	Red Lamp
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette Company.

(i) Find the optimum distribution for XYZ Tobacco Company.

(ii) Are there multiple optimum solutions ? If there are alternative optimum solutions, identify them.

(iii) Write the dual of the given transportation problem and use it for checking the optimum solution.

[Pubjab (M.B.A.,) 97]

Solution. (i) The given information is exhibited in the following table. Further, the problem is balanced by adding a dummy row and the cost element (of 7) in the given table corresponding to the cell (*A*—Bharat) is replaced by *M*, since the route is prohibited. Using VAM, the initial feasible solution is obtained and which, when tested for optimality is found to be optimum.

Initial Solution Optimal

	Bharat	Janta	Red Lamp	Supply	u_i
A	+(M − 13); (M) 13	+1; (10) 9	90 •; (5)	90	1
B	30 •; (12)	+1; (9) 8	20 •; (4)	50	0
C	0; (7) 7	80 •; (3) 3	+12; (11) −1	80	−5
D	40 •; (9)	20 •; (5)	+6; (7) 1	60	−3
Dammy	50 •; (0)	+4; (0) −4	+8; (0) −8	50	−12
Demand	120	100	110		
$v_j \rightarrow$	12	8	4		

Since opportunity cost in all the unoccupied cells is positive, initial solution shown in the above table is also an optimum solution. The total transport cost associated with this solution is :

$$5 \times 90 + 12 \times 30 + 4 \times 20 + 3 \times 80 + 9 \times 40 + 5 \times 20 = 1{,}59{,}000$$

(b) Since opportunity cost in cell (C, Bharat) $\Delta_{31} = 0$, there exists an alternative optimum solution :

$$x_{13} = 90, x_{21} = 30, x_{23} = 20, x_{31} = 40, x_{42} = 60, \text{total cost} = \text{Rs. } 1{,}59{,}000.$$

(c) The dual of the given problem is :

Maximize $Z = (90\,u_1 + 50\,u_2 + 80\,u_3 + 60\,u_4 + 50\,u_5) + (120\,v_1 + 100\,v_2 + 110\,v_3)$

subject to the constraints

$u_1 + v_1 \le M,$	$u_2 + v_3 \le 4$	$u_4 + v_2 \le 5$
$u_1 + v_2 \le 10$	$u_3 + v_1 \le 7$	$u_4 + v_3 \le 7$
$u_1 + v_3 \le 5$	$u_3 + v_2 \le 3$	
$u_2 + v_1 \le 12$	$u_3 + v_3 \le 11$	
$u_2 + v_2 \le 9$	$u_4 + v_1 \le 9$	

u_i, v_j unrestricted in sign ($i = 1, 2, 3$, and $j = 1, 2, 3, 4$). Using optimum values of u_i's and v_j's in the objective function, we get

$$90\,(1) + 50\,(0) + 80\,(-5) + 60\,(-3) + 50\,(-12) + 120 \times 12 + 100 \times 8 + 110 \times 4 = \text{Rs. } 1{,}59{,}000,$$

which is the same value as obtained earlier.

Example 20. *A company has four terminals U, V, W and X. At the start of a particular day 10, 4, 6 and 5 trailors are available at these terminals. During the previous night 13, 10, 6 and 6, trailors respectively were loaded at plants A, B, C and D. The company despatcher has come up with the costs between the terminals and plants as follows :*

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

Find the optimum sales distribution. **[Delhi (MCI) 2000; C.A., Nov. 97)**

Solution : The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Profit Matrix

Factory	Sales Centre			Supply
	1	2	3	
A	3	2	4	100
B	0	−1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The table representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (i.e., 5). The initial basic feasible solution can be found by using Vogel's method. Then obtain optimum solution.

[Ans. $x_{11} = 4$, $x_{13} = 6$, $x_{22} = 3$, $x_{24} = 1$, $x_{32} = 6$, $x_{44} = 5$, (Dummy) $x_{51} = 9$, $x_{52} = 1$. Min. transportation cost is Rs. 555.]

Example 21. *A multi-plant company has three manufactu- ring plants, A, B and C and two markets X and Y. Production cost at A, B and C is Rs. 1,500; 1,600 and 1,700 per piece respectively. Selling prices in X and Y are Rs. 4,400 and Rs. 4,700 respectively. Demands in X and Y are 3,500 and 3,600 pieces respectively. Production capacities at A, B and C are 2,000; 3,000 and 4,000 pieces respectively. Transportation costs are as shown in the adjacent table. Build a mathematical model.*

Plant	Market	
	X	Y
A	1,000	1,500
B	2,000	3,000
C	1,500	2,500

Solution. Here three plants differ in production cost and we are given the selling prices also. Therefore, our problem is to determine the schedule of production which may result in the maximum profit. The various profits per item are as shown in the adjacent table.

PROFIT MATRIX

Plant	Market	
	X	Y
A	1,900	1,700
B	800	100
C	1,200	500

The profit (selling price – production cost – transportation cost) data from plants to markets are shown below :

From A to $X: 4400 - 1500 - 1000 = 1900$; from A to $Y: 4700 - 1500 - 1500 = 1700$;

from B to $X: 4400 - 1600 - 2000 = 800$; and so on.

Further, total production at A, B and C plants is $2,000 + 3,000 + 4,000 = 9,000$ units while total requirement at X and Y is $3,500 + 3,600 = 7,100$ units. Hence this is an unbalanced transportation problem. By introducing a dummy market Z to receive an excess production of $9,000 - 7,100 = 1,900$ units, the complete relevant information is summarized in the following table :

	Market			Supply
	X	Y	Dummy	
Plant A	(1900)	(1700)	(0)	2000
Plant B	(800)	(100)	(0)	3000
Plant C	(1200)	(500)	(0)	4000
Demand	3500	3600	1900	9000

Let x_{ij} be the quantity to be transported from plant i $(i = 1, 2, 3)$ to market j $(j = 1, 2, 3)$.

Now the mathematical model based on the given data can be formulated as follows :

Maximize (total profit) $Z = 1900x_{11} + 1700x_{12} + 800x_{21} + 100x_{22} + 1200x_{31} + 500x_{32}$

Subject to the constraints

$$\left.\begin{aligned} x_{11}+x_{12}+x_{13} &= 2000 \\ x_{21}+x_{22}+x_{23} &= 3000 \\ x_{31}+x_{32}+x_{33} &= 4000 \end{aligned}\right\} \text{(Supply constraints)} \quad \left.\begin{aligned} x_{11}+x_{21}+x_{31} &= 3500 \\ x_{12}+x_{22}+x_{32} &= 7600 \\ x_{13}+x_{23}+x_{33} &= 1900 \end{aligned}\right\} \text{(Demand constraints)}$$

and $x_{ij} \geq 0$ ***for*** i ***and*** j.

Example 22. *A steel company has three open hearth furnaces and five rolling mills. Transportation cost (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table :*

Table 6·74

Mills

		M_1	M_2	M_3	M_4	M_5	Capacities (in quintals)
	F_1	4	2	3	2	6	8
Furnaces	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	3	14
Requirement (in quintals)		4	4	6	8	8	

[Meerut (Maths.)2003, 91]

What is the optimal shipping schedule ?

Solution. Since the total requirements of mills are 30 quintals and the total capacities of all furnaces are 34 quintals, the problem is of unbalanced type. Therefore, the problem can be modified as follows.

Step 1. ***(Modifying the given problem to balanced type).***

Since the capacities are *four* quintals more than the total requirements, consider a fictitious mill requiring *four* quintals of steel. Thus the modified (balanced) transportation cost matrix becomes :

Table 6·75

	M_1	M_2	M_3	M_4	M_5	M_f	Capacities
F_1	4	2	3	2	6	0	8
F_2	5	4	5	2	1	0	12
F_3	6	5	4	7	3	0	14
Required	4	4	6	8	8	4	34

Step 2. ***(To find the initial solution).***

Applying the Vogel's method in the usual manner, the initial solution is obtained as given below.

Table 6·76

	M_1	M_2	M_3	M_4	M_5	M_f
F_1		4(2)		4(2)		
F_2				4(2)	8(1)	
F_3	4(6)		6(4)			4(0)

This gives the transportation cost = 4(2) + 4(2)+ 4(2) + 4(2) + 8(1) + 4(6) + 6(4) + 4(0) = Rs. 80.

Step 3. ***(To test the initial solution for optimality).***

Since the total number of allocations is 7 (instead of 6 + 3 − 1 = 8), this is a degenerate basic feasible solution. Therefore, allocate an infinitesimal quantity Δ to empty cell (1, 1). Then, proceeding in the usual manner, following tables for testing the optimality of the solution are obtained.

Table 6·77

u_i and v_j

							u_i
	Δ (4)	• (2)		• (2)			0
				• (2)	• (1)		0
	• (6)		• (4)			• (0)	2
v_j	4	2	2	2	1	−2	

Table 6·78

$(u_i + v_j)$ for empty cells

•	•	2	•	1	−2	0
4	2	2	•	•	−2	0
•	4	•	4	3	•	2
4	2	2	2	1	−2	

Table 6·79

$d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells

•	•	1	•	5	8
1	2	3	•	•	2
•	1	•	3	0	•

Since all $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells are non-negative, the solution under test is optimal. Further, 0 in the cell (3,5) indicates that alternative solutions will also exist.

Example 23. *A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500, and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in Rs.) are given below:*

		To				
		D	E	*F*	*G*	*H*
	A	5	8	6	6	3
From	*B*	4	7	7	6	5
	C	8	4	6	6	4

Determine an optimum distribution for the company in order to minimize the total transportation cost. **[JNTU (B. Tech.) 2002, 98; AIMS 2002; Kerala B.Sc (Math.) 90]**

Solution. In this problem, the total warehouse requirements (= 2500 units) is greater than the total plant capacity (= 2200 units). Therefore, the problem is of unbalanced type. So introduce a dummy plant P having all transportation costs equal to zero and having the plant availability equal to (2500 − 2200) = 300 units. The modified transportation table is thus obtained as below :

		To					
		D	E	*F*	*G*	*H*	**Plant Capacity**
	A	5	8	6	6	3	800
From	*B*	4	7	7	6	5	500
	C	8	4	6	6	4	900
	P	0	0	0	0	0	300
Reuirements		400	400	500	400	800	

Using 'VAM' the following initial BFS is obtained :

	D	E	F	G	H	
A			500(6)		300(3)	800
B	400(4)			100 (6)	Δ (5)	500 + Δ = 500
C		400(4)			500(4)	900
P				300(0)		300
	400	400	500	400	800 + Δ = 800	

Since the number of basic cell allocations (= 7) is less than $m + n - 1$ (= 8), the solution is degenerate. To make the number of allocations equal to 8, introduce a negligible small positive quantity Δ in the independent cell (2, 5). Now test the current solution for optimality.

Starting Table

					u_i
+ (5) 2	+ (8) 3	500 − θ (6)	+ (6) 4	300 + θ (3)	− 2
400 (4)	+ (7) 5	− 1 (7) 8	100 + θ (6)	Δ − θ (5)	0
+ (8) 3	400 (4)	− 1 (6) 7	+ (6) 5	500 (4)	− 1
+ (0) − 2	+ (0) − 1	− 2 + θ √ (0) 2	300 − θ (0)	+ (0) − 1	− 6
v_j → 4	5	8	6	5	

Here θ = min [500, Δ, 300] = Δ. So enter the non-basic cell (4, 3) and leave the basic cell (2, 5).

First Iteration Table. Vacate the cell (2, 5) and ocupy the cell (4, 3).

					u_i
+ (5) 4	+ (8) 3	500 − θ (6)	0 (6) 6	300 + θ (3)	0
400 (4)	+ (7) 3	+ (7) 6	100 (6)	+ (5) 3	0
+ (8) 5	400 (4)	− 1 (6) 7	− 1 + θ √ (6) 7	500 − θ (4)	1
+ (0) − 2	+ (0) − 3	Δ + θ (0)	300 − θ (0)	+ (0) − 3	− 6
4	3	6	6	3	

Here min [500 − θ, 500 − θ, 300− θ] = 0 or 300 − θ = 0 or θ = 300.
So introduce the cell (3, 4) and drop the cell (4, 4) in the next iteration.

Second Iteration Table. *Vacate the cell (4, 4) and occupy the cell (3, 4).*

					u_i
+ (5) 3	+ (8) 3	200 − θ (6)	+ (6) 5	600 + θ (3)	− 1
400 (4)	+ (7) 4	0 (7) 7	100 (6)	+ (5) 4	0
+ (8) 4	400 (4)	− 1 + θ √ (6) 7	300 (6)	200 − θ (4)	0
+ (0) − 3	+ (0) − 3	300 (0)	+ (0) − 1	+ (0) − 3	− 7
v_j → 4	4	7	6	4	

Here min $[200-\theta, 200-\theta] = 0 \Rightarrow \theta = 200$.

So introduce the cell (3, 3) and drop the cell (1, 3) or (3, 5) in the next iteration.

Third Iteration Table. *Vacate the cell (1, 3) or (3, 5) and occupy the cell (3, 3).*

Optimum Table

					u_i
(5) + 4	(8) + 4	0 • (6)	(6) 0 6	800 • (3)	0
400 • (4)	(7) + 4	(7) + 6	100 • (6)	(5) + 3	0
(8) + 4	400 • (4)	200 • (6)	300 • (6)	(4) + 3	0
(0) + −2	(0) + −2	300 • (0)	(0) 0 0	(0) + −3	−6
v_j: 4	4	6	6	3	

Since all the net evaluations are non-negative, the optimum solution is :

$x_{13} = 0, x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200, x_{34} = 300, x_{43} = 300.$

The optimum transportation cost is given by

$z = 0\,(6) + 800\,(3) + 400\,(4) + 100\,(6) + 400\,(4) + 200\,(6) + 300\,(6) + 300\,(0) = \text{Rs. } 9200.$

Example 24. *The Bombay Transport Company has trucks available at four different sites in the following numbers :*

Site	:	*A*	*B*	*C*	*D*
No. of Trucks	:	5	10	7	3

Customers W, X, and Y require trucks as shown :

Customer	:	*W*	*X*	*Y*
No. of Trucks	:	5	8	10

Variable costs of getting trucks to the customers are :

From A to W → Rs 7, to X → Rs 3, to Y → Rs 6; From B to W → Rs 4, to X → Rs 6, to Y → Rs. 8; From C to W → Rs 5, to X → Rs 8, to Y → Rs. 4; From D to W → Rs 8, to X → Rs 4, to Y → Rs. 3.

Solve the above transportation problem.

Solution. Since the availability of trucks is greater than the requirement of trucks, the problem is of unbalanced type. Therefore, a dummy requirement column having all the transportation costs equal to zero has been introduced with $25 - 23\ (= 2)$ trucks as its requirement. The following balanced transportation table is then constructed :

	To: *W*	*X*	*Y*	*Z*	Available
From *A*	7	3	6	0	5
B	4	6	8	0	10
C	5	8	4	0	7
D	8	4	3	0	3
Requirement	5	8	10	2	

Now, using 'VAM', the initial basic feasible solution having transportation cost Rs 58 is obtained :

Initial BFS Table

	5(3)		
5(4)	3(6)	2(8)	
		5(4)	2(0)
		3(3)	

Starting Table. The solution under test is not optimum. The most negative cell evaluation is -4, so introduce the cell (2, 4).

Here, $\min[2-\theta, 2-\theta]=0 \Rightarrow \theta=2$.

Therefore, drop either the cell (2, 3) or (3, 4). The resulting solution degenerates.

					u_i
	+ ; (7) 1	5 • ; (3)	+ ; (6) 5	−1 ; (0) 1	−3
	5 • ; (4)	3 • ; (6)	$2-\theta$; (8)	√ −4 ; $+\theta$; (0) 4	0
	+ ; (5) 0	+ ; (8) 2	$5+\theta$; (4)	$2-\theta$; (0)	−4
	+ ; (8) −1	+ ; (4) 1	3 • ; (3) 1	+ ; (0) −1	−5
v_j	4	6	8	4	

First Iteration Table. Vacate the cell (3, 4) and occupy the cell (2, 4) with $\theta=2$.

Since all the net-evaluations are non-negative, the current solution is optimum. The optimum solution is given by $x_{12}=5, x_{21}=5, x_{22}=3, x_{23}=0, x_{24}=2, x_{33}=7, x_{43}=3$.

The optimum transportation cost is given by $z=5(3)+5(4)+3(6)+2(0)+7(4)+3(3)$ = Rs. 90.

Optimum Table

					$u_i \downarrow$
	+ ; (7) 1	5 • ; (3)	+ ; (6) 5	+ ; (0) −3	−3
	5 • ; (4)	3 • ; (6)	0 • ; (8)	2 • ; (0)	0
	+ ; (5) 0	+ ; (8) 2	7 • ; (4)	+ ; (0) −4	−4
	+ ; (8) −1	+ ; (4) 1	3 • ; (3)	+ ; (0) −5	−5
$v_j \rightarrow$	4	6	8	0	

Example 25. *Consider the following unbalanced transportation problem :*

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution.

		To 1	2	3	Supply
From	1	5	1	7	10
	2	6	4	6	80
	3	3	2	5	15
Demand		75	20	50	

Solution. In this problem, demand exceeds the supply. The problem is of unbalanced type. Therefore, introduce a '*dummy source*' whose transportation costs are given as 5, 3 and 2 respectively, and supply $145-105=40$. The modified transportation table is then constructed as follows :

		To 1	2	3	Supply
From	1	5	1	7	10
	2	6	4	6	80
	3	3	2	5	15
	4	5	3	2	40
Demand		75	20	50	

Using 'VAM', an initial basic feasible solution having the transportation cost Rs. 595 is obtained as given below :

Initial BFS Table

	10(1)		10
60(6)	10(4)	10(6)	80
15(3)			15
		40(2)	40
75	20	50	

Optimum Table

				u_i
	(5) 3 +	10 • (1)	(7) 3 +	−3
	60 • (6)	10 • (4)	10 • (6)	0
	15 • (3)	(2) 1 +	(5) 3 +	−3
	(5) 2 +	(3) 0 +	40 • (2)	−4
v_j	6	4	6	

Since all the net-evaluations are non-negative, the optimum solution is given by

$x_{12} = 10, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 15$ (allocation in dummy row is not considered)

The optimum transportation cost is given by $z = 0(1) + 60(6) + 10(4) + 10(6) + 15(3) =$ Rs. 515.

Example 26. *A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated required amount of gravel which will be needed at these construction projects :*

Project location	:	A	B	C
Weekly requirement (truck loads)	:	72	102	41

The company has three gravel pits in towns W, X and Y. The gravel required by the construction projects can be supplied by three pits. The amount of gravel which can be supplied by each pit is as follows :

Plant	:	W	X	Y
Amount available (truck loads)	:	76	82	77

The company has computed the delivery cost from each pit to each project site. These costs (in Rs) are shown in the following table :

Project Location

	A	B	C
W	4	8	8
X	16	24	16
Y	8	16	24

Schedule the shipment from each pit to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by pit capacities and project requirements. Find the minimum cost. Is the solution unique ? If it is not, find alternative schedule with the same minimum cost.

Solution. In this problem, total availability exceeds the total requirement of trucks. So the problem is of unbalanced type. Therefore, introduce a dummy project location D where all the transportation costs are zero and the requirement of the new project location is equal to the difference $(235 - 215) = 20$ units. Then the modified (balanced) transportation table becomes :

	A	B	C	D	Available
W	4	8	8	0	76
X	16	24	16	0	82
Y	8	16	24	0	77
Requirement	72	102	41	20	

Now using 'VAM', the initial BFS having the transportation cost of Rs. 2752 is obtained as below.

Initial BFS

	35(8)	41(8)		76
	62(24)		20(0)	82
72(8)	5(16)			77
72	102	41	20	

Starting Iteration. Since all the net-evaluations are not non-negaive, the initial B.F.S. is not optimum and can be improved further.

Here min $[62-\theta, 41-\theta]=0 \Rightarrow \theta=41$. So the cell (1, 3) should leave the basis and the non-basic cell (2, 3) must enter the basis.

				u_i ↓
+ (4) 0	35 + θ (8)	41 − θ (8)	+ (0) − 16	8
0 (16) 16	62 − θ (21)	−8 +θ √ (16) 24	20 • (0)	24
+ 72 • (8)	+ 5 • (16)	+ (24) 16	+ (0) − 8	16
v_j → − 8	0	0	− 24	

First Iteration Table. *Vacate the cell (1, 3) and occupy the cell (2, 3).*

Since all the $d_{ij} \geq 0$, the optimum solution is: $x_{12}=76, x_{22}=21, x_{23}=41, x_{31}=72, x_{32}=5$, ($x_{24}=20$ is dummy so it is neglected). This solution gives the minimum cost of Rs. 2424. Further, since $d_{21}=0$ for the empty cell (2, 1), *the solution obtained above is not unique.* If we again transfer $\theta=21$ units to empty cell (2, 1) and vacate the cell (2, 2), we at once obtain the alternative schedule : $x_{12}=76, x_{21}=21, x_{23}=41, x_{31}=51, x_{32}=26$ with the same minimum cost Rs. 2424.

				u_i ↓
+ (4) 0	76 • (8)	+ (8) 0	+ (0) − 16	− 16
0 (16) 16	21 • (24)	41 • (16)	20 • (0)	0
72 • (8)	5 • (16)	+ (24) 8	+ 0 − 8	− 8
v_j 16	24	16	0	

Example 27. *A company produces a small component for all industrial products and distributes it to five wholesalers at a fixed delivered price of Rs. 2·50 per unit. Sales forecasts indicate that monthly deliveries will be 3000, 3000, 10000, 5000, 4000 units to wholesalers 1, 2, 3, 4 and 5 respectively. The monthly production capacities are 5000, 10000 and 12500 at plants 1, 2 and 3 respectively. The direct costs of production of each unit are Rs. 1, Rs. 0·90 and Re 0·80 at plants 1, 2 and 3 respectively. The transportation costs of shipping a unit from a plant to a whole-saler are given below :*

Find how many components of each plant supplies to each wholesaler in order to maximize profit.

[CA (May) 2000]

Solution. Since the total capacity of plants is more than the supply to the wholesalers by a quantity 27,500 − 25,000 = 2,500 units, so the problem is of unbalanced type. Introduce a dummy wholesaler supplying 2500 units with all the transportation costs from the plants to this destination are assumed to be zero. Also the direct costs of production of each unit are given as Re 1, Re. 0·90 and Rs. 0·80 at plants 1, 2 and 3 respectively. The modified balanced transportation problem is now obtained as follows :

Wholesaler

		1	2	3	*4*	*5*
	1	·05	·07	·10	·15	·15
Plant	2	·08	·06	·09	·12	·14
	3	·10	·09	·08	·10	·15

Wholesaler

		1	2	3	4	5	6	
	1	1·05	1·07	1·10	1·15	1·15	0	5,000
Plant	2	0·98	0·96	0·99	1·02	1·04	0	10,000
	3	0·90	0·89	0·88	0·90	0·95	0	12,500
		3,000	3,000	10,000	5,000	4,000	2,500	

For simplicity in compulation, multiply all the transportation costs in the table by 100, and consider 100 units = 1 unit of items. So, simplified transportation table becomes :

		Wholesalers						
		1	2	3	4	5	6	Capacity
Plant	1	105	107	110	115	115	0	50
	2	98	96	99	102	104	0	100
	3	90	89	88	90	95	0	125
Supply		30	30	100	50	40	25	

Using 'VAM', the initial BFS having transportation cost of Rs 23,730 is obtained as follows :

25(105)					25(0)	50
5(98)	30(96)	25(99)		40(104)		100
		75(88)	50(90)			125
30	30	100	50	40	25	

Find the net-evaluations in the usual manner as shown in the following starting table.

							$\downarrow u_i$
	25 • (105)	+ (107) 103	+ (110) 106	+ (115) 108	+ (115) 111	25 • (0)	7
	5 • (98)	30 • (96)	25 • (99)	+ (102) 101	40 • (104)	+ (0) −7	0
	+ (90) 87	+ (89) 85	75 • (88)	50 • (90)	+ (95) 93	+ (0) −4	−11
$v_j \rightarrow$	98	96	99	101	104	7	

Since all the net-evaluations in the empty cells are non-negative, the optimum solution is given by

$$x_{11} = 2500, x_{21} = 500, x_{22} = 3000, x_{23} = 2500, x_{25} = 4000, x_{33} = 7500, x_{34} = 5000.$$

The optimum transportation cost is given by

$$z = 2500\,(1\cdot 05) + 500\,(\cdot 98) + 3000\,(\cdot 96) + 2500\,(\cdot 99) + 4000\,(1\cdot 04) + 7500\,(\cdot 88) + 5000\,(\cdot 90) = \text{Rs. } 23730.$$

Total 25000 units are supplied to the wholesalers at the fixed rate of Rs. 2·50 per unit,

∴ Total Sale = (25000 × 2.50) = Rs. 62,500.

Total production cost of three plants at the rate of Rs. 1, Rs. 0·90 and Rs. 0·80, respectively, becomes

= Rs. (5000 × 1 + 10,000 × ·90 + 12,000 × ·80) = Rs. 23,600.

Hence the net maximum profit to the manufacturer becomes = Rs. 62,500 – (Rs. 23,730 + Rs. 23,600)

= Rs. 15,170 .

Example 28. *A manufacturer of jeans is interested in developing an advertising compaign that will reach four different age groups. Advertising compaigns can be conducted through T.V., Radio and Magazines. The following table gives the estimated cost in paise per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely T.V., Radio and Magazines are 40, 30 and 20 millions respectively. Also the minimum desired exposures within each age group, namely 13–18, 19–25, 26–35, 36 and olders are 30, 25, 15 and 10 millions. The objective is to minimize the cost of attaining the minimum exposure level in each age group.*

Media	Age Groups			
	13–18	19–25	26–35	36 and older
T.V.	12	7	10	10
Radio	10	9	12	10
Magazine	14	12	9	12

(i) Formulate the above as a transportation problem, and find the optimal solution.

(ii) Solve this problem if the policy is to provide at least 4 million exposures through T.V. in the 13–18 age group and at least 8 million exposures through T.V. in the age group 19–25. **[C.A. (May) 91]**

Solution. (i) Formulation as a transportation problem.

Media	Age Groups				Max. exposure available (in millions)
	13–18	19–25	26–35	36 and older	
T.V.	12	7	10	10	40
Radio	10	9	12	10	30
Magazine	14	12	9	12	20
Minimum number of exposures required	30	25	15	10	80/90

Since this transportation problem is of unbalanced type, it can be made balanced by introducing a dummy category before applying *Vogel's Approximation Method.*

Initial Solution Table

Age Groups / Media	13–18	19–25	26–35	36 and older	Dummy	Max. exposure (in million)
T.V.		**25** (7)	**5** (10)	**10** (10)		40
Radio	**30** (10)					30
Magazine			**10**(9)		**10** (0)	20
Min. exposures required (million)	30	25	15	10	10	

The initial solution given by VAM is degenerate since there are only 6 allocations. We put a Δ in the least cost independent cell to proceed for optimality let $u_1 = 0$ and we calculate the remaining u_i and v_j's.

Age Groups

Media	13 – 18	19 – 25	26 – 35	36 and older	Dummy
T.V.	1 (12) 11	25 • (7)	5 − θ (10)	10 • (10)	√ −1 +θ (0) 1
Radio	30 • (10)	3 (9) 6	3 (12) 9	1 (10) 9	• Δ (0)
Magazine	4 (14) 10	6 (12) 6	10 + θ (9)	3 (12) 9	10 − θ (0)
$v_j \rightarrow$	11	7	10	10	1

Min. [5 − θ, 10 − θ] = 0 gives 5 − θ = 0 or θ = 5.

Improved Solution Table

						u_i ↓
	2 (12) 10	25 • (7)	1 (10) 9	10 • (10)	5 • (0)	0
	30 • (10)	2 (9) 7	3 (12) 9	0 (10) 10	Δ • (0)	0
	4 (14) 10	5 (12) 7	15 • (9)	2 (12) 10	5 • (0)	0
$v_j \rightarrow$	10	7	9	10	0	

Since all d_{ij}'s are non-negative, the improved solution is optimal.

Through T.V., 25 million people must be reached in the age-group 19–25 and 10 million people in the age group 36 & older.

Through Radio, 30 million people must be reached in the age group 13–18.

Through Magazines, 15 million people must be reached in the age grooup 26–35.

Total minimum cost of attaining the minimum exposure level is Rs. 71 lakhs.

Since $d_{24} = 0$, this solution is not unique. Alternative solutions also exist.

(ii) The required solution is given by

4 • (12)	25 • (7)	(10)	10 • (10)	1 • 0	40
26 • (10)	(9)	(12)	(10)	4 • 0	30
(14)	(12)	15 • (9)	(12)	5 • 0	20
30	25	15	10	10	

Total cost for this allotment is Rs. 71.8 lakhs.

Example 29. *A Company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarizes the data for this problem. The values in the body of the table represent net return on investment of one rupee upto the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year will grow to Rs. 1.90 by the end of the fourth year, yielding a net return of Rs. 0.90.*

Investment made at the beginning of year	Investment Type					Rupees available (in 000's)
	A	B	C	D	E	
	NET RETURN DATA					
1	0·80	0·90	0·60	0·75	1·00	500
2	0·55	0·65	0·40	0·60	0·50	600
3	0·30	0·25	0·30	0·50	0·20	750
4	0·15	0·12	0·25	0·35	0·10	800
Maximum Rupees Investment (in 000's)	750	600	500	800	1000	

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four year period.

Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon of four-year period. **[C.A. (May) 93]**

Solution. We observe that this transportation problem is of unbalanced type and it is a maximization problem. The step-by-step procedure is as follows :

Step 1. We balance the transportation problem by introducing a dummy year with net return 0 assigned to A, B, C, D and E.

Investment Type/Net Return Data

Years \ Type	A	B	C	D	E	Available Rs. (in 000's)
1	0·80	0·90	0·60	0·75	1·00	500
2	0·55	0·65	0·40	0·60	0·50	600
3	0·30	0·25	0·30	0·50	0·20	750
4	0·15	0·12	0·25	0·35	0·10	800
Dummy	0	0	0	0	0	1000
Max. Inv. (in 000's)	750	600	500	800	1000	3650

Step 2. Now convert the above *profit matrix* into a *loss matrix* by subtracting all the elements from the largest element Re 1·00 in the table.

Investment Type

Years \ Type	A	B	C	D	E
1	0·20	0·10	0·40	0·25	0
2	0·45	0·35	0·60	0·40	0·50
3	0·70	0·75	0·70	0·50	0·80
4	0·85	0·88	0·75	0·65	0·90
Dummy	1·00	1·00	1·00	1·00	1·00

Step 3. For convenience, we express the net loss data in above table in paise. Thereafter, we obtain the initial solution by VAM and apply transportation algorithm in the following table. Since all d_{ij}'s are not non-negative, we assign θ to most negative d_{ij} cell (5, 2).

					u_i ↓
20 (20) 0	Δ − θ (10)	50 (40) − 10	45 (25) − 20	500 + θ (0)	− 85
20 (45) 25	600 (35)	45 (60) 15	35 (40) 5	25 (50) 25	− 60
0 (70) 70	− 5 (75) 80	10 (70) 60	750 (50)	10 (80) 70	− 15
250 (85)	− 7 (88) 95	500 (75)	50 (65)	5 (90) 85	0
500 (100)	√θ (100) − 10 110	10 (100) 90	20 (100) 80	500 − θ (100)	15
v_j → 85	95	75	65	85	

Min [500 − θ, Δ − θ] = 0 gives θ = Δ.

Step 4. Putting θ = Δ, we get the revised solution and again apply optimality test in the following table.

					u_i ↓
20 (20) 0	10 (10) 0	50 (40) −10	45 (25) − 20	500 (0)	−85
10 (45) 35	600 (35)	35 (60) 25	25 (40) 15	15 (50) 35	−50
0 (70) 70	5 (75) 70	10 (70) 60	750 (50)	10 (80) 70	−15
250 (85)	3 (88) 85	500 (75)	50 (65)	5 (90) 85	0
500 (100)	Δ (100)	10 (100) 90	20 (100) 80	500 (100)	15
v_j → 85	85	75	65	85	

Since all d_{ij} are non-negative, the solution under test is optimal. The optimal allocation is given below.

Year	1	2	3	4	5	6
Investment type	E	B	D	A	C	D
Amount (in 000's)	500	600	750	250	500	50

The net return on investment for the planning horizon of four years is given by

$500 \times 1{\cdot}0 + 600 \times 0{\cdot}65 + 750 \times {\cdot}50 + 250 \times (0{\cdot}15) + 500 \times {\cdot}25 + 50 \times {\cdot}35 =$ Rs. 1445 thousands.

Example 30. *A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, the project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in the table.*

Auditor	Project		
	1 (Rs.)	2 (Rs.)	3 (Rs.)
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month. **[C.A. (May) 95]**

Solution. Formulation. The given problem can be put in the following tabular form of T.P.

		Project 1	2	3	Available
Auditor	1	1,200	1,500	1,900	160
	2	1,400	1,300	1,200	160
	3	1,600	1,400	1,500	160
	Required	130	140	160	

The given problem is of unbalanced type. So we introduce a dummy project to balance it, as follows :

		Project 1	2	3	Dummy	Available
Auditor	1	1,200	1,500	1,900	0	160
	2	1,400	1,300	1,200	0	160
	3	1,600	1,400	1,500	0	160
	Required	130	140	160	50	480

Here the problem is to maximize the total billing amount of the auditors. So first we convert this maximization problem into a minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff, *i.e.* Rs. 1900. Thus the minimization T.P. becomes :

		Project 1	2	3	Dummy	Available
Auditor	1	700	400	0	1,900	160
	2	500	600	700	1,900	160
	3	300	500	400	1,900	160
	Required	130	140	160	50	480

Now apply *VAM* for finding the initial feasible solution. Since it is a degenerate solution, we introduce a very small quantity Δ in the *least cost independent cell* (3, 3) and then apply optimality test in the usual manner. For convenience, we take figures of payoff's matrix in hundreds of rupees (Rs. 00's).

					$u_i \downarrow$
	8 (7) −1	3 (4) 1	160 • (0)	5 (19) 14	−4
	1 (5) 4	110 • (6)	2 (7) 5	50 • (19)	1
	130 • (3)	30 • (5)	Δ • (4)	1 (19) 18	0
$v_j \rightarrow$	3	5	4	18	

Since all d_{ij}'s for non-basic cells are positive, the initial solution obtained above by *VAM* is optimal.

The optimal allocation of projects to auditors and their billing amount is given below. Here an auditor may involve in more than one project as it is clear from the following :

Auditor	1	2	3	3
Project	3	2	1	2
Billing Amount (Rs.)	160× Rs. 1900 (Rs. 3,04,000)	110× Rs. 1300 (Rs. 1,43,000)	130 × Rs. 1600 (Rs. 2,08,000)	30 × Rs. 1400 (Rs. 42000)

Hence, the maximum total billing during the next month will be Rs. 6,97,000.

Example 31. *A particular product is manufactured in factories A, B, C and D; and is sold at centres 1, 2 and 3. The cost (in rupees) of product per unit and capacity (in kg.) per unit time of each plant is given below :*

Factory	Cost (Rs.) per unit	Capacity (kg.) per unit
A	12	100
B	15	20
C	11	60
D	18	80

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

Find the optimum sales distribution. **[C.A., Nov. 97]**

Solution. The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Profit Matrix

Factory	Sales Centre			Supply
	1	2	3	
A	3	2	4	100
B	0	−1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The *table* representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (*i.e.*, 5). The initial basic feasible solution is found by using Vogel's method.

Since the number of occupied cells are 6 which is one less than the required number $m + n - 1 = 7$, the solution is degenerate and after introducing an allocation of Δ to the least cost independent cell (*A*, 3), the initial solution is tested for optimality in *table* using MODI method.

Sales Centres

Factory	1	2	3	Supply	$u_i \downarrow$
A	100 • (2)	0 (3) 3	Δ • (1)	100	−1
B	0 (5) 5	20 • (6)	0 (4) 4	20	2
C	0 (1) 1	0 (2) 2	60 • (0)	60	−2
D	20 • (3)	60 • (4)	0 (2) 2	80	0
Dummy	+1 (5) 4	60 • (5)	0 (2) 2	60	1
Demand	120	140	60	320	
$v_j \rightarrow$	3	4	2		

Since there is no negative opportunity cost in the unoccupied cells in *table*, the solution is optimum. The total maximum profit associated with this solution is as follows :

$$100 \times 3 + 20 \times (-1) + 60 \times 5 + 20 \times 2 \times 60 \times 1 + 60 \times 0 = \text{Rs. } 680.$$

EXAMINATION PROBLEMS

1. Solve the following unbalanced transportation problem (symbols have their usual meanings).
   [IGNOU 2000]
   [**Ans.** $x_{12} = 5$, $x_{21} = 8$, $x_{23} = 4$, and min. cost = 23]

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	2	5	0	13
O_3	3	8	6	12
b_j	8	5	4	

2. Find an optimum basic feasible solution to the following transportation problem (use dummy destinations, if needed) :
   [**Ans.** $x_{12} = 400$, $x_{24} = 350$, $x_{25} = 50$, $x_{31} = 450$,
   $x_{33} = 200$ $x_{35} = 250$ and min cost = 6,100.]

	Destination					
Warehouse	5	4	8	6	5	600
	4	5	4	3	2	400
	3	6	5	8	4	1000
	450	400	200	250	300	

3. A company has three factories I, II, III and four warehouses 1, 2, 3 and 4. The transportation cost (in Rs.) per unit from each factory to each warehouse, the requirements of each warehouse, and the capacity of each factory are given below :
   Find the minimum cost transportation schedule.

	1	2	3	4	Capacity
I	25	17	25	14	300
II	15	10	18	24	500
III	16	20	8	13	600
Requirement	300	300	500	500	

   [**Ans.** $x_{14} = 300$, $x_{21} = 200$, $x_{22} = 300$, $x_{33} = 500$, $x_{34} = 100$ and min cost = 15,500.]

4. Consider the following transportation problem with the cost matrix :

	1	2	3	4	5	6	Available
1	5	1	5	8	9	7	30
2	4	3	1	9	2	2	40
3	2	1	3	2	8	2	10
4	1	0	2	8	6	3	110
Required	50	20	10	35	15	50	

   (i) Determine a shipping scheme by any method.
   (ii) Test the above solution for optimality.
   (iii) If the above solution is not optimal, find a better solution (you need not to find the optimal solution)
   [**Ans.** (i) $x_{13} = 10$, $x_{14} = 10$, $x_{25} = 15$, $x_{26} = 25$, $x_{34} = 10$, $x_{41} = 50$, $x_{42} = 20$, $x_{44} = 15$, $x_{46} = 25$.
   (ii) Not optimal.
   (iii) $x_{14} = 20$, $x_{23} = 10$, $x_{25} = 15$, $x_{26} = 15$, $x_{34} = 10$, $x_{44} = 50$, $x_{42} = 20$, $x_{44} = 5$, $x_{46} = 35$.]

5. Consider the transportation problem with the following cost matrix :

Origins	Destinations				Available
	P	Q	R	S	
A	4	6	8	13	50
B	13	11	10	8	70
C	14	4	10	13	30
D	9	11	13	8	50
Required	25	35	105	20	—

   (i) Determine a shipping scheme by the north-west corner rule.
   (ii) Test the above solution for optimality.
   (iii) If the above solution is not optimal, find a better one.
   [**Ans.** (i) $x_{11} = 25$, $x_{12} = 25$, $x_{22} = 10$, $x_{23} = 60$, $x_{33} = 30$, $x_{43} = 15$, $x_{44} = 20$.
   (ii) No.
   (iii) $x_{11} = 25$, $x_{12} = 5$, $x_{13} = 20$, $x_{23} = 70$, $x_{32} = 30$, $x_{43} = 15$, $x_{44} = 20$; and min. cost = 1465.]

6.13 TIME-MINIMIZING TRANSPORTATION PROBLEMS

In time-minimizing transportation problems, the objective is to minimize the time of transportation rather than the cost of transportation. For example in military, the time of supply is considered more valuable than the cost of transportation and, therefore, it is always prefered to minimize the total time of supply and not the cost. Such problems are often encountered in emergency services like military services, hospital management, fire services, etc.

In fact, the time-minimization transportation problems are similar to the cost-minimization transportation problems, except that the unit transportation cost c_{ij} is replaced by the unit time t_{ij} required to transport the items from ith origin to jth destination.

The *feasible plan* (*initial basic feasible solution*) for this problem can also be obtained by using any of the methods discussed in **Sec. 6·8.** Now, if T_j is the largest transportation time associated with the jth feasible plan, then we have to find such plan that gives minimum (T_j).

The *step-by-step* procedure for the solution of such problems is given below.

6·13–1 Solution Procedure for Time-Minimization T.P.

Step 1. First, find an initial basic feasible solution by using any of the methods discussed in **Sec. 6·8.** Enter the solution at the centres of the basic cells.

Step 2. Compute T_j for this basic feasible solution and cross out all the non-basic cells for which $t_{ij} \geq T_j$.

Step 3. Now construct a loop for the basic cells corresponding to T_j in such a way that when the values at the centre of the cells are shifted around, the value at this cell tends towards (not-necessarily) zero and no variable becomes zero. If no such closed path is possible, the solution under test is optimal, otherwise go to ***step 2.***

Step 4. Repeat the procedure until an optimum basic feasible solution is attained.

Q. 1. What are the least time transportation problem ?
2. What are the areas, where the time minimization problems are applied ?
3. State the characteristics of a "loop" of a least-time transportaion problems.
4. Give the solution procedure for solving the least-time transportation problems.

6·13–2. Computational Demonstration of Solution Procedure

The computational procedure is best explained by the following example.

Example 32. *If the matrix elements represent the time, solve the following transportation problem* :

		To				
		D_1	D_2	D_3	D_4	Available
	O_1	10	0	20	11	15
From	O_2	1	7	9	20	25
	O_3	12	14	16	18	5
Required		12	8	15	10	45

Solution. **First Iteration :**

Step 1. Using *Vogel's Approximation Method,* find an initial basic feasible solution as given in the ***Table 6.80*** The numbers written in the down-left corners of each cell represent the corresponding times.

Step 2. The times for this feasible plan are : $t_{12} = 0$, $t_{14} = 11$, $t_{21} = 1$, $t_{23} = 9$, $t_{33} = 16$, and $t_{34} = 18$.

$\therefore \quad T_1 = \max. [0, 11, 1, 9, 16, 18] = 18.$

Obviously, all the shipments for this plan are to be completed within the time $T_1 = 18$. So we cross-out the cell (1, 3) and (2, 4) because $t_{13} > T_1$ and $t_{34} > T_1$.

Table 6·80

	D_1	D_2	D_3	D_4
O_1		8 (0)		7 (11)
O_2	12 (1)		13 (9)	
O_3			2 (16)	3 (18)

Table 6.81

				u_i
(10)	8 − (0)	(20) ✕	7 + (11)	0
(1)	(7)	13 (9)	(20) ✕	0
(12)	+ (14)	2 (16)	3 − (18)	7
v_j → 1	0	9	11	

Step 3. Now the closed path (loop) for the cell (3, 4) for which $t_{34} = T_1 = 18$ is shown in *Table 6.81*. It is evident from ***Table 6.81*** that only 3 units can be shifted around.

Second Iteration :

Step 1. The revised feasible plan is shown in ***Table 6.82.***

Table 6.82

(10)	8 (0)	(20) ✕	10 (11)
12 − (1)	(7)	13 + (9)	(20) ✕
+ (12)	3 (14)	2 − (16)	(18) ✕

Step 2. Here $T_2 = \max\{t_{12}, t_{14}, t_{21}, t_{22}, t_{32}, t_{33}\} = \max.\{0, 11, 1, 9, 14, 16\} = 16$

As all the shipments for this feasible solution are completed within time $T_2 = 16$, we cross-out the cell (3, 4) also since $t_{34} > T_2$.

Step 3. The closed loop starts from cell (3, 3) as shown in ***Table 6.82*** above. Clearly, only 2 units can be shifted around.

Third Iteration :

Step. 1. The revised feasible plan is shown in ***Table 6·83***.

Table 6.83

(10)	5 • (0)	(20) ×	10 • (11)
− 10 (1)	+ (7)	15 • (9)	(20) ×
+ 2 (12)	3 − (14)	(16) ×	(18) ×

Step 2. Here $T_3 = \max\{t_{12}, t_{14}, t_{21}, t_{23}, t_{31}, t_{32}\} = \max\{0, 11, 1, 9, 12, 14\} = 14$.
As all the shipments for this feasible plan are completed within time $T_3 = 14$, we cross-out the cell (3, 3) also, since $t_{33} > T_3$.

Step 3. Since $t_{32} = T_3 = 14$, The closed loop starts from the cell (3, 2) as shown in ***Table* 6.83.** . Clearly, 3 units can be shifted around.

Fourth Iteration :

Step 1. The next revised feasible plan is shown in ***Table* 6.84**

Table 6.84

(10)	5 • (0)	(20) ×	10 • (11)
7 • (1)	3 • (7)	15 • (9)	(20) ×
5 • (12)	(14) ×	(16) ×	(18) ×

Step 2. Here $T_4 = \max\{t_{12}, t_{14}, t_{21}, t_{22}, t_{23}, t_{31}\} = \max\{0, 11, 1, 7, 9, 12\} = 12$.
As all the shipments for this feasible plan are completed within time $T_4 = 12$, we cross-out the cell (3, 2) also since $t_{32} > 12$.

Step 3. Now we connot form any closed loop without increasing the present minimum shipping time. Hence the feasible plan at this stage is optimal.
Thus, all the shipments can be made within 12 time units.

EXAMINATION PROBLEM

1. Solve the following transportation problem, the matrix represents the times t_{ij} :

To

		D_1	D_2	D_3	D_4	Available
	O_1	6	7	3	4	5
From	O_2	7	9	1	2	7
	O_3	6	5	16	7	8
	O_4	18	9	10	2	10
Required		10	5	10	5	30

[**Ans.** Total shipment time is 9 units. Details of plan are :
$x_{11} = 2$ with $t_{11} = 6$; $x_{13} = 3$ with $t_{13} = 3$; $x_{23} = 7$ with $t_{23} = 1$; $x_{31} = 8$ with $t_{31} = 6$; and $x_{42} = 5$ with $t_{42} = 9$.]

6.14 TRANSHIPMENT PROBLEMS

Definition. *A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a* ***transhipment problem.***

13·14.1 Main Characteristics of Transhipment Problems

Following are the main characteristics of transhipment problems :

1. The number of sources and destinations in the transportation problem are m and n respectively. But in transhipment problems, we have $m + n$ sources and destinations.
2. If S_i denotes the ith source and D_j denotes the jth destination, then commodity can move along the route $S_i \to D_i \to D_j$, $S_i \to S_j \to D_i \to D_j$, $S_i \to D_i \to S_j \to D_j$, or in various other ways. Clearly, transportation cost from S_i to S_i is zero and the transportation costs from S_i to S_j or S_i to D_i do not have to be symmetrical, *i.e.*, in general, $S_i \to S_j \neq S_j \to S_i$.
3. While solving the transhipment problem, we first obtain the optimum solution to the transportation problem, and then proceed in the same manner as in solving the transportation problems.
4. The basic feasible solution contains $2m + 2n - 1$ basic variables. If we omit the variables appearing in the $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

Q. 1. Explain transhipment problem.
2. What are the main characteristics of a transhipment problem ?
3. Explain the method of solving the transhipment problem.
4. Indicate how a transhipment problem can be solved as a transportation problem. **[Garhwal 97]**
5. What is transhipment problem ? Show how a transhipment problem can be modelled as a transportation problem. **[Delhi (M.B.A) 95]**
6. Define a Transhipment problem. How does it defer from a transportation problem. How is it solved ?

6·14·2. Computational Demonstration of Solution Procedure

The computational procedure for solving transhipment problems is best explained by the following example.

Example 33. *Consider the following transhipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule :*

	S_1	S_2	D_1	D_2	
S_1	0	1	3	4	5
S_2	1	0	2	4	25
D_1	3	2	0	1	
D_2	4	4	1	0	
			20	10	30

Solution. Step 1. *(To get modified transportation problem).* In the transhipment problem, each given source and destination can be considered a source or a destination. If we now take the quantity available at each of the sources D_1 and D_2 to be zero and also at each of the destinations S_1 and S_2 the requirement to be zero, then to have a supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as *'buffer stock'* is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to Σa_i or Σb_j. In our problem, the buffer-stock comes-out to be 30 units.

Modified Table 13.85

	S_1	S_2	D_1	D_2	Available
S_1	0	1	3	4	35
S_2	1	0	2	4	55
D_1	3	2	0	1	30
D_2	4	4	1	0	30
Required	30	30	50	40	

Step 2. ***(To find initial solution of modified problem)***

By adding 30 units of commodity to each point of supply and demand, an initial basic feasible solution is obtained in ***Table 6.86*** by using *Vogel's Approximation method.*

Starting Table 6.86

					a_i
	(0) 30 •	(1)	(3)	(4) 5 •	35
	(1)	(0) 30 •	(2) 20 •	(4) 5 •	55
	(3)	(2)	(0) 30 •	(1)	30
	(4)	(4)	(1)	(0) 30 •	30
b_j	30	30	50	40	

Step 3. ***(To apply optimality test)***

The variables u_i $(i = 1, 2, 3, 4)$ and v_j $(j = 1, 2, 3, 4)$ have been determined by using successively the relations $u_i + v_j = c_{ij}$ for all the basic (occupied) cells. These values are then used to compute the net-evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic (empty) cells. Clearly d_{34} $(= -1)$ is the only negative quantity. Hence an unknown quantity θ is assigned to this cell (3, 4). After identifying the loop, we find that θ = 5 and that the cell (2, 4) leaves the basis (*i.e.*, becomes empty).

Table 6.87

					u_i
	(0) 30 •	(1) 1, 0 + 0	(3) 1, 0 + 2	(4) 5 •	0
	(1) 1, 0 + 0	(0) 30 •	(2) 20 + θ •	(4) 5 − θ •	0
	(3) 5, −2 + 0	(2) 4, −2 + 0	(0) 30 − θ •	(1) −1, + θ √, −2 + 4	−2
	(4) 8, −4 + 0	(4) 8, −4 + 0	(1) 3, −4 + 2	(0) 30 •	−4
v_j	0	0	2	4	

Step 4. ***Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then, again test the optimality of the revised solution.***

Since all the current net evaluations are non-negative, the current solution is an optimum one. It is shown in Table 12.88. The minimum transportation cost is:

$$z^* = 5 \times 4 + 25 \times 2 + 5 \times 1 = 75.$$

and the optimum transportation route is as shown below.

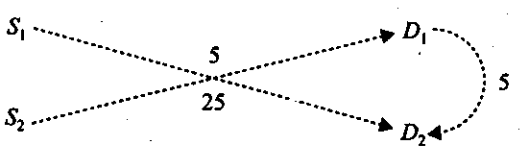


Table 6.88

				u_i
30 • (0)	0 (1) −1	0 (3) 3	5 • (4)	0
2 (1) −1	30 • (0)	25 • (2)	1 (4) 3	−1
6 (3) −3	4 (2) −2	25 • (0)	5 • (1)	−3
8 (4) −4	7 (4) −3	2 (1) −1	30 • (0)	−4
v_j: 0	1	3	4	

EXAMINATION PROBLEMS

1. Given the following data, find the optimum transportation routes:

	S_1	S_2	D_1	D_2	Capacity
S_1	0	2	2	1	8
S_2	1	0	2	3	3
D_1	2	2	0	2	
D_2	1	3	2	0	
Demand			7	4	

[**Ans.** $(S_1, D_1) = 4$ units, $(S_1, D_2) = 4$ units, $(S_2, D_1) = 3$ units; min. cost = 18.]

2. The unit cost of transportation from site i to site j are given below. At site $i = 1, 2, 3$, stocks of 150, 200, 170 units respectively, are available. 300 units are to be sent to site 4 and rest to site 5. Find the cheapest way of doing this:

	1	2	3	4	5
1	–	3	4	13	7
2	1	–	2	16	6
3	7	4	–	12	13
4	8	3	9	–	5
5	2	1	7	5	—

[**Ans.** $x_{14} = 130$, $x_{15} = 20$, $x_{25} = 200$, $x_{34} = 170$; and min. cost = 5070.]

6.15 GENERALIZED TRANSPORTATION PROBLEM

A transportation problem is called generalized transportation problem if the constraints of the general transportation problem are expressed as

$$\sum_{j=1}^{n} a_{ij} x_{ij} = a_i \quad (1 \le i \le m), \quad \sum_{i=1}^{m} b_{ij} x_{ij} = b_j \quad (1 \le j \le n) \quad \text{and} \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

with $x_{ij} \ge 0$, $(i = 1, 2, ..., m, j = 1, 2, ..., n)$. In the transportation problem, linearly independent equations were $m + n - 1$ in number, and the fact led to the simplicity with which an initial basic feasible solution could be found and tested for optimality. This advantage disappears in the present case. Fortunately, it is still possible to establish an algorithm on the similar lines as that for transportation problem, but is not so simple in practice. Recently, various algorithms have been developed for the solution of generalized problems by many research workers in this field, which are too extensive to be included here.

Q. 1. Indicate how a transhipment problem can be solved as a transportation problem.

2. M-sources have s_i, $i = 1, \ldots, M$, units of a commodity while demand at retail shop j is d_j, $j = 1, \ldots, N$. Profit per unit supplied from source i to shop j is p_{ij}, $i = 1, \ldots, M$; $j = 1, \ldots, N$. Indicate how the profit-maximizing problem with the above data can be converted to an equivalent cost-minimizing problem, stating clearly any more assumptions one can make in this regard.
3. What is meant by balanced transportation problem ? Give a method of solving transportation problem with capacities.
4. Explain in detail, any one method for solving a transportation problem. Would you recommend this method to solve an assignment problem.
5. What is degeneracy in transportation problems ? How transportation problem is solved when demand and supply are not equal.
6. State the Transportation Problem in general terms and explain the problem of degeneracy. How does one overcome it ? **[I.A.S. (Maths.) 90]**
7. What is the total number of constraint equations in a general transportation model with m sources and n destinations ? How many of these are independent ? Justify your answer. Name the methods for obtaining an initial basic solution. Which is the best one, and why ? Also describe it. **[Meerut (OR) 2003]**

EXAMINATIONS REVIEW PROBLEMS

1. Solve the following transportation problem :

		Consumers A	B	C	Available
Suppliers	I	6	8	4	14
	II	4	0	8	12
	III	1	2	6	5
Required		6	10	15	31

[Hint. Use 'VAM' to find an initial BFS and prove it to be optimal.]
[Ans. $x_{12} = 14$, $x_{21} = 6$, $x_{22} = 5$, $x_{23} = 1$, $x_{32} = 5$, min. cost = Rs. 143].

2. A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 160, 150 and 190 units respectively. Monthly warehouse requirements are 80, 90, 110 and 160 units respectively. Unit shipping costs (in Rs.) are given in the table. Determine the optimum distribution for this company to minimize shipping costs.

		To D	E	F	G
From	A	42	48	38	37
	B	40	49	52	51
	C	39	38	40	43

[Hint. Unbalanced type problem. Degeneracy may occur. Use 'VAM' for initial BFS and improve it for optimality.]
[Ans. $x_{14} = 160$, $x_{21} = 80$, $x_{22} = 10$, $x_{32} = 80$, $x_{33} = 110$, min. cost = Rs. 17050. Alternate solution exists].

3. Given below the unit costs array with supplies a_i, $i = 1, 2, 3$; and demands b_j, $j = 1, 2, 3, 4$. Find the optimum solution to this Hitchcock problem.

		Sink 1	2	3	4	$a_i \downarrow$
Source	1	8	10	7	6	50
	2	12	9	4	7	40
	3	9	11	10	8	30
	$b_j \rightarrow$	25	32	40	23	120

[Ans. $x_{11} = 25$, $x_{12} = 2$, $x_{13} = \Delta$, $x_{25} = 40$, $x_{14} = 23$, $x_{32} = 30$, min. cost = Rs. 840]

4. A company has three factories F_i ($i = 1, 2, 3$) from which it transports the product to four warehouses W_j ($j = 1, 2, 3, 4$). The unit cost of production at the three factories are Rs. 4, 3, 5 respectively. Given the following information on unit costs of transportation, capacities at the three factories and requirements at the four warehouses. Find the optimum allocations.

Unit cost of Production		Transportation cost W_1	W_2	W_3	W_4	Available
F_1	4	5	7	3	8	300
F_2	3	4	6	9	5	500
F_3	5	2	6	4	5	200
Required		200	300	400	100	1,000

5. A company has four factories F_1, F_2, F_3 and F_4 and four warehouses, W_1, W_2, W_3 and W_4. The warhouses are located at varying distances from the factories, from where the supplies are transported to them; the transportation costs from the factories to the warehouses thus naturally vary from Rs. 2 to Rs. 6 per unit and the company desires to minimize these transportation costs. In the form of a matrix, the costs from the factories to the warehouses are as shown in the table.

Warehouses

		W_1	W_2	W_3	W_4	Capacity
Factories	F_1	2	3	4	5	400
	F_2	3	2	3	4	500
	F_3	4	3	3	4	600
	F_4	6	4	4	5	700
Required		700	600	500	400	

Assign factory capacities to warehouse requirements so as to minimize the costs of transportation by making use of the technique of linear programming.

[Ans. $x_{13} = 300$, $x_{21} = 100$, $x_{22} = 300$, $x_{24} = 100$, $x_{31} = 100$, $x_{33} = 100$, min. cost = Rs. 3200].

6. A company has factories A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 250, 300 and 400 units respectively for regular production. If overtime production is utilized, factories A and B can produce 50 and 75 additional units respectively at overtime incremental costs of Rs. 4 and Rs. 5 respectively. The current warehouse requirement are 200, 225, 275 and 300 units respectively. Unit transportation costs (in Rs.) from factories to the warehouses are as below.

To

		D	E	F	G
From	A	11	13	17	14
	B	16	18	14	10
	C	21	24	13	10

Determine the optimum distribution for this company to minimize costs.

[Ans.

	D	E	F	G	H	a_i
A	11	13	17	14	0	250
B	16	28	14	10	0	300
C	21	24	13	10	0	400
A'	15	17	21	18	0	50
B'	21	23	19	15	0	75
b_j	200	225	275	300	75	

]

7. A manufacturer must produce a certain product in sufficient quantity to meet contracted sales in the next four months. The production facilities available for this product are limited, but by different amounts in respective months. The unit cost of production also varies accordingly to the facilities and personnel available. The product may be produced in one month and then held for sale in a later month, but an estimated storage cost of Re. 1 per unit per month. No storage cost is incurred for goods sold in the same month in which they are produced. There is presently no inventory of this product and none is desired at the end of 4 months. Given the following table show how much to produce in each of four months in order to minimize total cost.

Month	Contracted sales (in units)	Maximum production (in units)	Unit cost of production (Rs.)	Unit storage cost per month (Rs.)
1	20	40	14	1
2	30	50	16	1
3	50	30	15	1
4	40	50	17	1

Formulate the problem as a transportation problem and hence solve it.

[Ans.

	1	2	3	4	5	a_i ↓
1	14	15	16	17	0	40
2	0	16	17	18	0	50
3	0	0	15	16	0	30
4	0	0	0	17	0	50
$b_j \rightarrow$	20	30	50	40	30	

]

8. Company has four warehouses *a, b, c, d*. It is required to deliver a product from these warehouses to three customers *A*, *B*, and *C*. The warehouses have the following amounts in stock :

Warehouse	:	*a*	*b*	*c*	*d*
No. of units	:	15	16	12	13

and the customer's requirements are :

Customer	:	*A*	*B*	*C*
No. of *units*	:	18	20	18

The table below shows the costs of transporting one unit from warehouse to the customer.

	a	b	c	d
A	8	9	6	3
B	6	11	5	10
C	3	8	7	9

Solve the problem.

[**Hint.** Use 'VAM' to find initial BFS and prove it to be optimal.]

[**Ans.** $x_{12} = 5$, $x_{14} = 13$, $x_{22} = 8$, $x_{23} = 12$, $x_{31} = 15$, $x_{32} = 3$, min. cost = Rs. 331]

9. Goods have to be transported from factories F_1, F_2, F_3, to warehouses W_1, W_2, W_3, and W_4. The transportation costs per unit, capacities of the factories and requirements of the warehouses are given in the following table. Find the distribution with minimum cost.

	W_1	W_2	W_3	W_4	Capacity
F_1	15	24	11	12	5000
F_2	25	20	14	16	4000
F_3	12	16	22	13	4000
Requirement	3000	2500	3500	4000	

[**Ans.** $x_{13} = 2{,}500$, $x_{14} = 2{,}500$, $x_{23} = 1000$, $x_{21} = 3000$, $x_{32} = 2500$, $x_{34} = 1500$ and min. $z = 1{,}67{,}000$.]

10. A company has four Factories from which it ships its product units to four warehouses W_1, W_2, W_3 and W_4 which are the distribution centres. Transportation costs per unit between various combinations of factories (F_1, F_2, F_3 and F_4) and warehouses are as:

	W_1	W_2	W_3	W_4	Available
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
F_4	52	64	55	61	220
Required	200	320	250	210	

Find the transportation schedule which minimizes the distribution cost.

[**Hint.** Find the initial solution by VAM as $(F_1, W_2) = 60$, $(F_1, W_3) = 30$, $(F_1, W_4) = 50$ $(F_2, W_2) = 260$ $(F_3, W_1) = 200$ $(F_3, W_4) = 160$, $(F_4, W_3) = 220$ units.

and prove it to be optimum. Since d_{33} will be zero, there will exist alternative solutions also.]

11. Solve the following transportation problem using *north-west corner rule* for initial feasible solution.

A company has 3 Plants P_1, P_2, P_3 each producing 50, 100 and 150 units of a similar product. There are five warehouses W_1, W_2, W_3, W_4 and W_5 having demand of 100, 70, 50, 40 and 40 units respectively.

The cost of sending a unit from various plants to the warehouses differs as given by the cost matrix below. Determine a transportation schedule so that cost is minimized.

	W_1	W_2	W_3	W_4	W_5	a_i
P_1	20	28	32	55	70	50
P_2	48	36	40	44	25	100
P_3	35	55	22	45	48	150
b_j	100	70	50	40	40	300/300

[**Ans.** $(P_1, W_1) = 40$, $(P_1, W_2) = 10$, $(P_2, W_2) = 60$, $(P_2, W_5) = 40$
$(P_3, W_1) = 60$, $(P_3, W_3) = 50$, $(P_3, W_4) = 40$, min. cost = Rs. 9240].

12. A transportation problem with 3 sources and 4 destinations has :

$C_{11} = 2$, $C_{12} = 3$, $C_{13} = 11$, $C_{14} = 7$, $C_{21} = 1$, $C_{22} = 0$, $C_{23} = 6$, $C_{24} = 1$, $C_{31} = 5$, $C_{32} = 8$, $C_{33} = 15$, $C_{34} = 9$
$S_1 = 6$, $S_2 = 1$, $S_3 = 10$, $D_1 = 7$, $D_2 = 5$, $D_3 = 3$, $D_4 = 2$

Solve this T.P. **[Meerut (Maths.) 96]**

13. A state has four government hospitals *A*, *B*, *C* and *D*. Their monthly requirements of medicines, etc. are met by four distribution centres *X*, *Y*, *Z* and *W*. The data in respect of a particular item *via-a-vis* availabilities at the centres, requirements at the hospitals and distribution cost per unit (in paise) is given in the following table :

Distribution centre	Hospital				Availability
	A	*B*	*C*	*D*	
X	44	84	84	80	2,000
Y	92	30	64	80	12,000
Z	32	100	96	72	5,000
W	80	36	120	60	6,000
Requirement	8,000	8,000	6,000	3,000	25,000

Determine the optimum distribution. **(Delhi (M.B.A.) Nov. 98)**

[**Hint :** First find initial solution by VAM.

Ans. $x_{11} = 100$, $x_{23} = 200$, $x_{33} = 450$, $x_{41} = 400$, $x_{53} = 200$, $x_{62} = 350$, $x_{71} = 100$, $x_{72} = 50$, $x_{73} = 150$.]

14. Four gasoline dealers *A*, *B*, *C* and *D* require 50, 40, 60 and 40 KL of gasoline respectively. It is possible to supply these from locations 1, 2 and 3 which have 80, 100 and 50 KL respectively. The cost (in Rs.) for shipping every KL is shown in the table below :

Location

		A	*B*	*C*	*D*
Location	1	7	6	6	6
	2	5	7	6	7
	3	8	5	8	6

Determine the most economical supply pattern. **(Jammu (M.B.A.) 96)**

[**Hint :** Add a dummy dealer to make the problem balanced.

Ans. $x_{13} = 10$, $x_{14} = 30$, $x_{21} = 50$, $x_{23} = 50$, $x_{32} = 40$, $x_{34} = 10$ and min. cost = Rs. 1,050.]

15. Priyanshu Enterprise has three factories at locations *A*, *B* and *C* which supplies three warehouse located at *D*, *E* and *F*. Monthly factory capacities are 10, 80 and 15 units respectively. Monthly warehouse requirements are 75, 20 and 50 units respectively. Unit shipping costs (in Rs.) are given in the following table :

		Warehouse		
		D	*E*	*F*
Factory	*A*	5	1	7
	B	6	4	6
	C	3	2	5

The penalty costs for not satisfying demand at the warehouses *D*, *E* and *F* are Rs. five, Rs. three, and Rs. two per unit respectively. Determine the optimum distribution for Priyanshu, using any of the known algorithms write the dual of this problem. **(Delhi (M.B.A.) March 99)**

[**Hint :** Demand (145) > Supply (105). Add dummy source with supply 40 and transportation costs 5, 3 and 2 for destination 1, 2 and 3 respectively.

Ans. $x_{12} = 0$, $x_{21} = 60$, $x_{22} = 10$, $x_{23} = 10$, $x_{31} = 15$ and $x_{43} = 40$, min. cost = Rs. 514. Penalty for transporting 40 units to destination 3 at the cost of Rs. 2 per unit is Rs. 80.]

16. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depto 3; and 440,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	Company 2	Company 3
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum schedule. **[Delhi (M.B.A.) Nov. 97]**

[**Hint :** Supply (1485000) > Demand (1100000), add dummy column with demand of 385000 eithers of oil.

Ans. $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, min. transportation cost = Rs. 51,700. Thus required cost = Rs. 515 + 80 + Rs. 595.]

17. A departmental store wishes to purchase the following types of sarees :

Types of sarees :	A	B	C	D	E
Quantity :	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities mentioned below (all types of sarees combined) :

Manufacturer :	W	X	Y	Z
Total quantity :	300	250	150	200

The store estimates that its profit per saree will vary with the manufacturer as shown below :

Manufacturer	Saree				
	A	B	C	D	E
W	275	350	425	225	150
X	300	325	450	175	100
Y	250	350	475	200	125
Z	325	275	400	250	175

Use transportation technique to determine how the orders should be placed ? What is the maximum profit ?

(Punjabi (M.B.A.) 96)

[**Hint :** First add dummy column with demand 125 sarees and subtract all elements of the profit matrix from the highest element 475.

Ans. $x_{12} = 25$, $x_{14} = 50$, $x_{15} = 200$, $x_{16} = 25$, $x_{21} = 150$, $x_{27} = 100$, $x_{32} = 75$, $x_{33} = 75$, $x_{44} = 200$.]

18. A manufacturer has distribution centres at Delhi Kolkata and Chennai. These centres have available 30, 50 and 70 units of product. His four retail outlets require the following number of units.

A, 30; B, 20; C, 60; D, 40;

The transportation cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets			
	A	B	C	D
Delhi	10	7	3	6
Kolkata	1	6	7	3
Chennai	7	4	5	3

Determine the minimum transportation cost. **[IAS (Main) 2001]**

19. A company has factories at four different places which supply to warehouses *A*, *B*, *C*, *D* and *E*. Monthly factory capacities are 200, 175, 150 and 325 respectively. Unit shipping costs in rupees are given below :

		Warehouses				
		A	B	C	D	E
Factories	1	13	—	31	8	20
	2	14	9	17	6	10
	3	25	11	12	17	15
	4	10	21	13	—	17

Shipping from 1 to B and from 4 to D is not possible. Determine the optimum distribution to minimize the shipping cost.

[Banglore 2002]

MODEL OBJECTIVE QUESTIONS

1. The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
(a) the solution be optimal.
(b) the rim conditions are satisfied.
(c) the solution not be degenerate.
(d) all of the above.

2. The dummy source or destination in a transportation problem is added to
(a) satisfy rim conditions.
(b) prevent solution from becoming degenerate.
(c) ensure that total cost does not exceed a limit.
(d) none of the above.

3. The occurrence of degeneracy while solving a transportation problem means that
(a) total supply equals total demand. (b) the solution so obtained is not feasible.
(c) the few allocations become negative. (d) none of the above.

4. An alternative optinal solution to a minimization transportation problem exists whenever opportunity cost corresponding to unused route of transportation is:
(a) positive and greater than zero. (b) positive with at least one is equal to zero.
(c) negative with at least one is equal to zero. (d) none of the above.

5. One disadvantage of using North-West Corner Rule to find initial solution to the transportation problem is that
(a) it is complicated to use. (b) it does not take into account the cost of transportation.
(c) it leads to a degenerate initial solution. (d) all of the above.

6 The solution to a transportation problem with m - rows (supplies) and n- columns (destinations) is feasible if number of positive allocations are
(a) $m + n$. (b) $m \times n$. (c) $m + n - 1$. (d) $m + n + 1$.

7. The calculation of opportunity cost in the MODI method is analogous to a
(a) $z_j - c_j$ value for non-basic variable columns in the simplex method.
(b) value of a variable in x_B-column of the simplex method.
(c) variable in the B-column in the simplex method.
(d) none of the above.

8. An unoccupied cell in the transportation method is analogous to a
(a) $z_j - c_j$ value in the simplex method. (b) variable in the B-column in the simplex method.
(c) variable not in the B-column in the simplex method. (d) value in the X_B-column in the simplex method.

9. If we were to use opportunity cost value for an unused cell to test optimality, it should be
(a) equal to zero. (b) most negative number. (c) most positive number. (d) any value.

10. During an iteration while moving from one solution to the next, degeneracy may occur when
(a) the closed path indicates a diagonal move.
(b) two or more occupied cells are on the closed path but neither of them represents a corner of the path.
(c) two or more occupied cells on the closed path with minus sign are tied for lowest circled value.
(d) either of the above.

11. Consider the following statements :
1. an optimal solution 2. an initial feasible solution 3. Vogel's approximation solution
Of these statements :
(a) 1 alone is correct. (b) 2 alone is correct. (c) 3 alone is correct. (d) 2 and 3 are correct.
[IES (Mech.) 1993]

12. In a 6 × 6 transportation problem, degeneracy would arise, if the number of filled spots were
(a) equal to thirty six. (b) more than twelve. (c) equal to twelve. (d) less than eleven.
[IES (Mech.) 1993]

13. The solution of a transportation model (of dimension $m \times n$) is said to be degenerate if it has
(a) exaclty $(m + n - 1)$ allocations. (b) fewer than $(m + n - 1)$ allocations.
(c) more than $(m + n - 1)$ allocations. (d) $m + n$ allocation. **[IES (Mech.) 1995]**

14. A solution is not a basic feasible solution in a transportation problem if after allocations :
(a) there is no closed loop. (b) there is a closed loop.
(c) total number of allocations is one less than the sum of the number of sources and destrinations.
(d) there is degeneracy. **[IES (Mech.) 1996]**

15. When there are 'm' rows and 'n' columns in a transportation problem, degeneracy is said to occur when the number of allocation is
(a) less than $m + n - 1$. (b) greater than $m + n - 1$. (c) equal to $m + n - 1$. (d) less than $m - n - 1$.
[IES 1997]

16. In transportation problem, the materials are transported from 3 plants to 5 warehouses. The basic feasible solution must contain exaclty which one of the following allocated cells ?
(a) 3 (b) 5 (c) 7 (d) 8 **[IES 1998]**

Answers

1. (a)	2. (a)	3. (b)	4. (b)	5. (b)	6. (c)	7. (a)	8. (c)	9. (b)
10. (c)	11. (a)	12. (d)	13. (b)	14. (a)	15. (a)	16. (d)		

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